# EUCLID'S ELEMENTS,

OR

# SECOND LESSONS IN GEOMETRY,

IN THE ORDER OF

SIMSON'S AND PLAYFAIR'S EDITIONS

ADAPTED TO THE USE OF

ADVANCED LEARNERS AND PRIVATE STUDENTS.

BY D. M'CURDY,

AUTHOR OF THE "CHART OF GEOMETRY" AND "FIRST LESSONS."

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# PREFACE.

To bring the Elements of Geometry into general use, is the design of this volume, and of the "Chart of Geometry" and "First Lessons" which precede it.

There was once a competition between certain persons to be the first who should see the risen sun; and the prize was awarded to him who turned his face westward: because there the sun's effects were first discovered, in gilding towers, and battlements, and the mountain's brow. To ascertain the existence of geometry by its effects, let us turn from books to the community, and the obvious defect will meet us in every department of life. Few citizens know what these things mean, or what their use.

A question then arises, "Should this be so?" The regrets of thousands prove the contrary. The learning to read and write is a mere preparation to receive instruction: after which, the learner should take hold of the properties of things, and examine them in detail, beginning with the most general, and therefore the most useful. But are there any properties more general than those of magnitude, figure, and motion? There are none: the attribute of number itself is not more general, and it is certainly less expedient as a branch of study. The cherished motto, "A place for everything," evinces the necessity of geometry in all the schools. The magnitude and figure of *everything*, and of the space to contain it, as well as the law of motion and the momentum of force which conveys it to the place, are certainly more worthy of consideration than the mere fact that it counts one.

It is obvious from the perfection in which the elements of geometry have been handed down to us, that the Greeks taught these elements in all their schools; that geometry was to them what arithmetic has been to us, namely, the groundwork of public instruction. See, then, the effects of this practice in their works of art, their architecture, their sculpture, their literature, their philosophy, the spread of their language, the respect paid to them by the Romans after the conquest of Macedonia. The advantages of a right education to a people are incalculable.

An opinion is widely entertained, namely, that algebra should have priority of geometry in the order of study. This reverses the natural order of the studies; for what is algebra but a method of managing arithmetic and geometry? It prepares certain general formulæ, and teaches the reduction of them, by transposition, substitution, elimination, &c. This may be done as an envelope is prepared for a letter:—the letter must be enclosed, or the envelope is of no use. It requires knowledge of the relations of numbers, of magnitudes, or figures, as the case may be, to dispose in an equation the *data* and *quæsita* of a proposition. Any simple problem will illustrate the absolute dependence of algebra upon geometry, and how preposterous the idea of giving it priority. Let us seek the ordinate of a circle from its relation to the diameter ($d$) and the abscissa ($x$): the formula is this, $\sqrt{dx-x^2}$=ordinate. Now, in view of the diagram, these relations are plainly seen; but without it, the formula would be as abstruse to the juvenile capacity as the Chinese language.

After this manner, a multitude of useful theorems are lost to the community, from two causes: one is the neglect of geometry; the other is, that the theorems are placed under the unfriendly umbrage of algebraic symbols; and this latter calls itself the "modern improvements of science!" But that is no improvement in science which precludes the general diffusion of knowledge. There is therefore a want of order, as well as a defect, in the parts which constitute the elementary education.

Mathematical reasoning is conducted according to two *methods;* one is called the method of *analysis*, or *resolution;* the other is called the method of *synthesis*, or *composition*. Algebra adopts the former of these; separating the known from the unknown parts of a general proposition; representing number and magnitude by symbols, and descending, by a succession of equivalent propositions, from the most complex to the simplest form. Geometry adopts the synthetic method, which begins at the simplest elements, and proceeds, by easy steps, to the more complex combinations; and it is proper to remark, that this alone is the process by which the known and the unknown parts of a general proposition can be distinguished.

It is obvious that the cost of books, on this plan, and

the labor of the study, are both greatly reduced; and the method by previous recitations of the text, which is the exclusive object of the book of "First Lessons," and which is continued in the present volume, will enable teachers less proficient to use the work, without apprehension of error or loss of time. There is, moreover, an assurance of success connected with this system, which no other has given, or can give. What is the deficiency with all of us? Why have we not more men entitled to degrees? The reason is this: We are not masters of the elements of science; we cannot call up at will the proofs of many propositions: we wanted this culture when we were a young people; we want it still, and should take heed lest we entail the same want on our posterity.

In allusion to the peculiarities of this work, it is unnecessary to be specific. Let the book be examined on its merits, and candidly compared with volumes of two or three times the size; no defect, it is believed, will be found in it, and it will not seem to be redundant. The order of Euclid has been preserved, because it has never been excelled; but the repetitions which swell other editions and perplex the learner, are here obviated by the plan of previous recitations.

The demonstrations of the fifth book have been simplified exceedingly; and in one or two instances reduced from two octavo pages to a few lines: not by the substitution of symbols for words, but by a new definition of ratio, and a slight alteration in the method of compounding ratios. Nevertheless, care has been taken to introduce nothing which could not be directly employed in drawing out the properties of proportionals according to the rigor of the Euclidian geometry. The change consists in fixing definitely the value of ratio agreeably to its general use in the mathematics.

Mental arithmetic and mental algebra are sublimated abstractions, not affording the proper exercise for the juvenile mind. Children are entitled to the use of their senses before they are required to reason: because, to reason is to combine, compare, digest, and dispose in order the materials received through the senses.

Now, arithmetic furnishes an infinite variety of series, the terms of which have varied relations to each other:

but the law of each series must be known, in order to bring those relations into view. The relations of geometry, however, are presented to the eye;—they are realities; the business of the world in miniature: it reasons from certain *data*, and furnishes the best model of reasoning in things less certain.

It was not design that brought arithmetic to occupy the exclusive ground it has held in the schools. Children could always repeat several terms of the natural series of numbers, and merchants could always sum and sever their gains and losses: in addition to this, the Moorish system of notation had come into use, carrying the series beyond any assignable limit. At the time of the revival of letters, a basis was thus provided for the study of arithmetic; and to read, write, and cipher, was esteemed accomplished scholarship. No regard was had to the mechanical operations of the world, or to the laws of the universe. Since that time, arithmetic has been compiled and re-modelled into a thousand forms, and algebra superadded; but geometry, which should be, at least, co-ordinate with arithmetic, has been neglected, or left to the speculative scholar. Yet these elements belong to the operatives—to the employments of men; they arm men with the skill and force of Nature, and imbue them with her wise designs; they verify the adage that "knowledge is power."

There is a solemn voice in the natural truths of geometry, which calls upon School Officers, Professors, and Teachers, to sow the seed, to scatter it *broadcast* over this national husbandry, to sacrifice the distinctions of learning to the perpetuation of a wise popular government, through the medium of an efficient elementary education. And this voice would be obeyed, and this motive deemed sufficient, if something great were required to be done; but when it is merely to put into the hands of children the text of Euclid, to be read and recited, will they not say, "Where is the use of it?" Where then is the use of scattering so much good wheat over the fields? Is it all lost? No! the stoutest doubter expects twenty to one of the same kind. By the same rule, therefore, we may have twenty Euclids to one from the Common Schools and Academies.

*New York, March*, 1846.

# SECOND LESSONS IN GEOMETRY.

## BOOK I.

### *Definitions.*

1. GEOMETRY is the science which treats of the similarity, equality, difference and proportions of magnitudes and of figures of extension.

2. A point is a position, or station in a line, at the extremities of a finite line, also at the meeting and intersection of lines: but it is not the measuring unit, nor any part of the measure of a line.

Cor. Hence, points are by position, central, angular, sectional, or extreme.

3. A finite line is that of which the extreme points are given.

Note. There are two classes of lines; namely, straight and curved: of curves there are several species; but the circle alone will be here considered. Lines have lengths, but no other dimensions.

4. A straight line is the path of a point, without curve or angle.

Cor. Two straight lines cannot meet and part and meet again: they cannot have a common segment; but, meeting in two points, or coinciding in part, they shall coincide in all their length.

Note. Straight lines have certain relations to one another from their position; namely, *perpendicular*, *meeting*, *insisting*, *parallel*, and *intersecting*. One line is *perpendicular* to another when it makes the adjacent angles equal, or when it *pends*, or hangs upon the other as the plumbline upon the level: lines *meet* when they touch and do not cut one another; one line *insists* upon another when it stands upon a point in the other: one line is *parallel* to another when it is in the same plane with the other, and at all points equidistant from it: lines *intersect* one another when they pass through the same point.

5. A *circular* line is the path of a moving point about a stationary one, at the same uniform distance from it.

6. A superficies is the upper or outside face—the surface: it has two dimensions—length and breadth: it is bounded by a line, or lines; and the intersection of two superficies is a line.

7. A plane is a superficies described by the lateral motion of a straight line; or, by its rotary motion about one of its extreme points.

8. A plane angle is the rotary declination of one straight line from another, about a stationary point in which they meet.

NOTE. Angles are, by position, *adjacent or opposite*, *interior or exterior*, *vertical or alternate;* by magnitude, they are *acute, right, or obtuse;* the acute and obtuse are called *oblique* angles. *Salient* and *re-*

*entrant* angles are the outward angles in fortifications; the former are greater and the latter less than half the compass of the angular point.

9. An acute angle is any declination of two straight lines, smaller than one-fourth of the compass of the angular point.

10. A right angle is the declination of two straight lines to one-fourth of the compass of the angular point.

11. An obtuse angle is any declination of two straight lines *greater than one-fourth* but *less than half* the compass of the angular point.

Cor. When the declination of two straight lines is equal to half the compass of the point in which they meet, they form no angle, but are in one straight line.

12. A plane figure is any form of a superficies,—it is bounded by at least three straight lines.

13. A circle is a plane figure enclosed by a uniformly curved line, called the circumference.

14. The centre of a circle is a point within it, equidistant from every point of the circumference.

15. A radius is any straight line drawn from the centre to the circumference of a circle: therefore all radii of the same or equal circles are equal to one another.

16. A diameter *of a circle* is a straight line drawn through the centre to the circumference on either side;—of a *parallelogram* is that straight line which joins opposite angles.

17. A semicircle is a figure contained by the diameter and half the circumference.

18. An arc is any part of the circumference of a circle,—a chord is the straight line which joins the extremities of the arc.

19. A sector is the figure contained by two radii and an arc.

20. A segment of a circle is a part cut off by a chord,—a segment of a line is a part cut off or distinct.

21. Rectilineal figures are those which are enclosed by straight lines: trilaterals have three sides; quadrilaterals, four, &c.

22. Trilateral figures, or triangles, have six parts; namely, three sides and three angles; from which they take the following names:

23. An equilateral triangle has three equal sides.

24. An isosceles triangle has two equal sides.

25. A scalene triangle has three sides unequal.

26. An acute-angled triangle has its three angles acute.

27. A right-angled triangle has one right angle.

28. An obtuse-angled triangle has one obtuse angle.

29. Quadrilateral figures, or quadrangles, are contained by four straight lines: of this kind are the square and oblong, the rhombus and rhomboid, the trapezoid and trapezium.

30. A square has four equal sides, and four right angles.

31. An oblong, or rectangle, has four right angles, and its opposite sides equal and parallel.

32. A rhombus has four equal sides, of which the opposite are parallel; and four oblique angles, of which the opposite are equal.

33. A rhomboid has its opposite sides and angles equal, and all its angles oblique.

34. A trapezoid has two of its opposite sides parallel.

35. All other quadrilateral figures are called trapezia.

36. Multilateral figures, or polygons, are contained by five or more sides; but the triangle and square may be included in the series. They are called regular when they are equilateral.

NOTE. The series of regular polygons are equiangular: it may be said to begin with the triangle, and end with the circle.

The Greeks named the regular polygons from their angles, viz:

| | | | |
|---|---|---|---|
| A trigon has | three | equal | angles. |
| A tetragon has | four | " | " |
| A pentagon has | five | " | " |
| A hexagon has | six | " | " |
| A heptagon has | seven | " | " |
| An octagon has | eight | " | " |
| A nonagon has | nine | " | " |
| A decagon has | ten | " | " |
| An undecagon has | eleven | " | " |
| A dodecagon has | twelve | " | " |

A trisdecagon has thirteen equal angles; a quindecagon, fifteen; and so upwards to infinity, which may be represented by the circle.

37. The perimeter is the sum of the lines which bound a figure: and it may be remarked, although it will be hereafter proved, that under equal perimeters, regular polygons contain greater areas than figures contain whose sides are unequal.

Also, the perimeters of regular polygons being equal, that perimeter which has the greater number of sides, contains the greater area; and the circle contains the greatest area within equal bounds.

38. A theorem is a proposition which requires to be proved.

39. A problem is a proposition requiring something to be done

40. A corollary is an inference from a previous demonstration.

41. A scholium is a remark explanatory of the preceding subject.

## POSTULATES, OR PETITIONS.

The geometer requires permission to move freely in space, viz:

1. To draw a straight line from one point to another.
2. To produce a terminated straight line.
3. To describe a circle about any point with any radius.

## AXIOMS, OR MAXIMS.

1. Things which are equal to the same are equal to each other.
2. Add equals to equals, the sums will be equal.
3. Take equals from equals, the remainders will be equal.
4. Add equals to unequals, the sums will be unequal.
5. Take equals from unequals, the remainders will be unequal.
6. Doubles of the same are equal to each other.
7. Halves of the same are equal to each other.
8. Things which coincide, or fill the same space, are equal to each other.
9. The whole is equal to all its parts, and greater than its part.
10. All right angles are equal to one another; and so are all angles measured by equal arcs to equal radii.

11. Two intersecting straight lines cannot both be parallel to the same straight line, or to each other.

## *Illustration of the Definitions.*

The angular point is marked by a letter, and when three or more lines meet in a point, their extremities are lettered; and in naming the angle, the letter at the angular point is read between the other two.

Fig. 1.

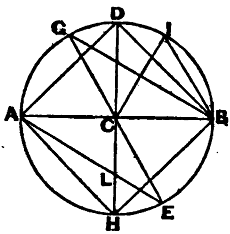
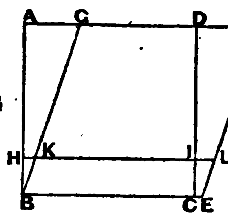



Fig. 2.




2. Point, central, sectional, or extreme.
3. Line, straight, or curved.
4. Straight line CI, Fig. 1.
" Perpendicular CD to AB.
" Insisting lines CD, CI, &c.
" Parallels AD to BH.
" Intersecting lines EG, DH.
6. Superficies.
7. Plane.
8. Plane angle ACG.
9. Acute do BCI.
10. Right do BCD.
11. Obtuse do BCG.
12. Plane figure.
13. Circle ADBH.
14. Centre C
15. Radius CB, CE, &c.
16. Diameter of circle or square, AB, DH.
17. Semicircle ADB.
18. Arc and chord AH.
19. Sector DCI.
20. Segment ADG.
21. Rectilineal figures.
22. Trilaterals.
23. Equilateral triangle BCI.
24. Isosceles do ACH.
25. Scalene do ALH.
26. Acute angled do BCI.
27. Right angled do BCD.
28. Obtuse angled do BCG.

29. Quadrilaterals, Fig. 2.
30. Square } rectangles { ABCD
31. Oblong } rectangles { AHID
32. Rhombus BEFG
33. Rhomboid GKLF.
34. Trapezoid BCDG.
35. Trapezia.
36. Polygons.

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*Abbreviations used in the following work:—*

P, after the number of the proposition, stands for problem; Th. for theorem; p., after recite, for proposition; ax., axiom; cor., corollary; def., definition; pos., postulate; hyp., hypothesis. The first figure refers to the proposition; the latter, to the book.

# SECOND LESSONS IN GEOMETRY.

## BOOK FIRST.

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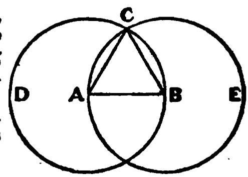
### *Propositions.*

**1 P.** To describe an equilateral triangle upon a given straight line (AB).



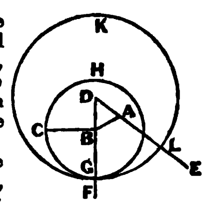

Construction. From the extreme points, A and B, with the radius AB, describe the circles ACE, BCD (*a*), intersecting each other in the point C: join C to A and B (*b*).

Demonstration. ABC is the required triangle (*c*): for AB one of its sides, is also a radius of each of the circles (*d*); and the sides AC, BC, are also radii, each equal to AB: therefore the three sides are equal to one another (*e*), and the triangle is equilateral; and it is described upon AB, the given straight line, which was to be done.

Recite (*a*), postulate 3; (*b*), post. 1; (*c*), definition 23; (*d*), def. 15; (*e*), axiom 1.

Corollary. An isosceles triangle may be constructed by joining the extreme points of two radii.

**2 P.** From a given point (A), to draw a straight line equal to a given straight line (BC).

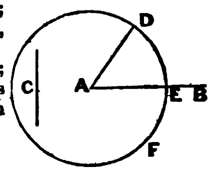



Constr. Join the points A, B, by the line AB (*a*), upon which describe the equilateral triangle ABD (*b*); produce the equal sides DA, DB to E, F (*c*); upon the centre B, with the radius BC, describe the circle CGH; also, upon the centre D, with radius DG, describe the circle GKL (*d*): AL is the required line.

Dem. From the equal radii DG, DL, take the equals DB, DA; the remainders BG, AL, are equal (*e*); but BG, BC, are equal radii; therefore AL is equal to BC (*f*); and it is drawn from the given point A, which was to be done.

Recite (*a*), post. 1; (*b*), prop. 1; (*c*), post. 2; (*d*), post. 3; (*e*), ax. 3; (*f*), ax. 1.

**3 P.** From the greater (AB), of two given straight lines, to cut off a part equal to the less (C).

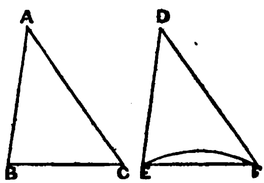



Constr. Draw the line AD equal to C, (*a*); and from the centre A, with the distance AD, describe the circle DEF (*b*).

Dem. AD and AE are equal radii (*c*); but AD is made equal to C; therefore AE is equal to C (*d*); and it is a part cut off from AB, which was to be done.

Recite (*a*), prop. 2; (*b*), post. 3; (*c*), def. 15; (*d*), ax. 1.

**4 Th.** If two triangles (ABC, DEF), have two sides (AB, AC) of the one, equal to two sides (DE, DF), of the other, each to each; and have likewise the angles (A and D) contained by those sides, equal; their third sides (BC and EF) shall also be equal; and their areas shall be equal; also their other angles—namely (B to E and C to F) those to which the equal sides are opposite.



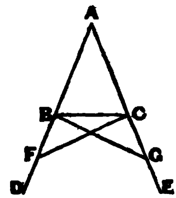

Because the three given parts are adjacent in each triangle, and equal, each to each; therefore, the parts BA, A and AC, may be applied to the parts ED, D and DF, so that they shall fill the same space (*a*), and that the points B, A and C, shall severally coincide with the points E, D and F; hence the line BC shall fall on the line EF, and be equal to it; otherwise, falling above or below the line EF, two straight lines would enclose a space, which is impossible (*b*). Therefore, also the angles at B and C shall be equal to those at E and F, respectively; and the areas of the two triangles shall coincide and be equal (*a*).

Wherefore, if two triangles, &c. Q. E. D.

Recite (*a*), ax. 8; (*b*), cor. to def. 4.

Note. This proposition is of very general use in the elements: it helps to demonstrate the 5th and 47th of the first book, and many others. It proves the triangles ABG, ACF, in the next diagram, to be equal; as also the triangles FBC, GCB. In this case, as in all others, two things are compared by means of a third; the third thing here taken is a portion of space, to which each of the triangles is applied.

**5 Th.** The angles (ABC, ACB), at the base of an isosceles triangle, are equal to one another; and if the equal sides (AB, AC,) be produced (to D and E), the angles (BCE, CBD,) below the base (BC) shall be equal.




Constr. In BD take any point F; and from AE, the greater, cut off a part, AG, equal to AF ($a$); then from the equals AF, AG, take the equals AB, AC, the remainders, BF and CG, will be equal ($b$); join BG, CF ($c$).

Dem. The two triangles ABG, ACF, are equal; having two sides, AB, AG, of the one, equal to two sides, AC, AF, of the other, each to each; and the angle A is common to both: therefore the bases, BG and CF, are equal, and likewise the angles ABG, ACF; as also the angles at F, G ($d$).

The two triangles BCG, CBF, are also equal: for it is shown above, that FC, FB, and the angle F, in the one, are severally equal to GB, GC, and the angle G, in the other; and the base BC is common to both; therefore the remaining angles are equal, each to each, to which the equal sides are opposite; viz. CBG to BCF, and FBC to GCB, which are the angles below the base ($d$).

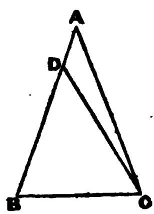
Again. From the equal angles, ABG, ACF, take the equals CBG, BCF, the remainders, ABC, ACB, are equal, which are the angles at the base ($e$).

Wherefore, the angles, &c. Q. E. D.

Recite ($a$), p. 3; ($b$), def. 24, ax. 3; ($c$), post. 1; ($d$), p. 4; ($e$), ax. 3.

Corollary. Hence every equilateral triangle is also equiangular.

6 Th. If two angles (B, C) of a triangle (ABC), be equal to one another, the subtending sides (AC, AB) of the equal angles shall be equal to one another.




Constr. For if AB be not equal to AC, it must be less than it, or greater. Let AB be the greater, and from it cut off a part BD equal to AC, the less ($a$); and join CD ($b$).

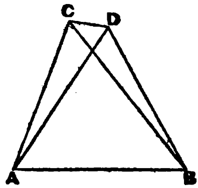
Dem. Because, in the triangles DBC, ACB, DB is equal to AC, and BC is common to both; therefore, the two sides DB, BC, are equal to the two AC, CB, each to each; and the angles DBC, ACB, are equal: therefore the bases DC, AB, are equal; and the triangle DBC is equal to the triangle ACB ($c$), the less to the greater, which is absurd. Therefore AB is not unequal to AC, but equal to it.

Wherefore, if two angles, &c. Q. E. D

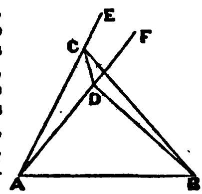
Recite ($a$), prop. 3; ($b$), post. 1; ($c$), prop. 4.

Cor. Hence every equiangular triangle is also equilateral.

**7 Th.** Upon the same base (AB), and on one side of it, two triangles cannot have their sides (AC, AD) equal, which terminate in one extremity of it, and likewise their sides (BC, BD), which terminate in the other extremity.




Join CD: then, in the case in which the vertex of each triangle is without the other; because AC and AD are equal, the angles ACD, ADC, are also equal (*a*). But the angle BCD is less than the angle ACD, therefore less than ADC, and still less than BDC. Again, because CB is equal to DB, the angles BCD, BDC, are equal: but BCD has been proved to be much less than BDC; both equal and less is impossible.



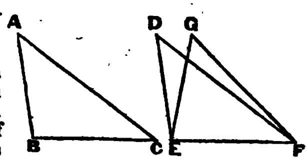

And, in the case of one of the vertices, D, being within the other triangle ACB, produce AC, AD, to E, F; therefore, because ACD is an isosceles triangle, the angles ECD, FDC, beyond the base, are equal (*b*); but the angle BCD is less than the angle ECD, therefore less than FDC, and still less than BDC. Again, because BC and BD are equal, the angles BDC, BCD, are also equal: but BCD has been proved to be less than BDC, which is impossible.

The case in which the vertex of one triangle is upon the side of another, needs no demonstration.

Therefore, upon the same base, &c. Q. E. D.

Recite (*a*), prop. 5, and (*b*), prop. 5.

**8 Th.** If two triangles have two sides (AB, AC) of the one, equal to two sides (DE, DF) of the other, each to each, and have likewise their bases (BC, EF) equal; the angles (A, D) contained by the corresponding sides, shall be equal to one another.



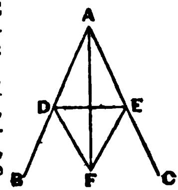

Demonstration. Apply the triangle ABC to DEF; so that the point B fall on E, and the base BC fall upon its equal EF; then the point C shall coincide with F, and the sides BA, AC, fall on ED, DF, each upon each; for, if the bases coincide, but the sides fall in another direction, as on G, then upon one side of the base EF, there can be two triangles, with equal sides, which terminate in either extremity of the base; which is impossible (*a*). Wherefore, if the bases

coincide, the sides shall also coincide; and, consequently, the angles BAC, EDF, contained by those sides, shall be equal. Q. E. D.

Recite (*a*) prop. 7.

**9 P.** To bisect a given rectilineal angle (BAC); that is, to divide it into two equal angles.

Constr. In AB take any point D; make AE equal to AD (*a*); join DE (*b*), and upon it describe an equilateral triangle DEF (*c*); join AF: the straight line AF bisects the angle BAC.

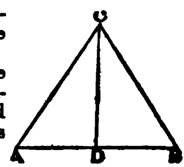



Dem. Because AD is made equal to AE, and AF is common to the two triangles DAF, EAF, and the bases DF, EF, were made equal: therefore the angle DAF is equal to EAF (*d*); that is, the rectilineal angle BAC is bisected by the straight line AF; which was to be done.

Recite (*a*), prop. 3; (*b*), post. 1; (*c*), prop. 1; (*d*), prop. 8.

**10. P.** To bisect a given finite straight line (AB); that is, to divide it into two equal parts.

Constr. Upon AB describe an equilateral triangle ABC (*a*), and bisect the angle ACB by the straight line CD (*b*).

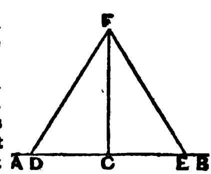



Argument. AC is made equal to BC, the angle ACD to BCD, and CD is common to the two triangles ACD, BCD: therefore, the bases AD and BD are equal, and AB is cut into two equal parts in the point D. Which was to be done.

Recite (*a*), prop. 1; (*b*), prop. 9.

**11 P.** To draw a straight line (CF), at right angles to a given straight line (AB), from a given point (C), in the same.

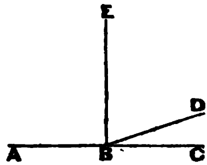
Construction. In AC take any point D, and make CE equal to CD (*a*); upon DE describe the equilateral triangle DFE (*b*); join FC (*c*).




Argument. The sides CD, CF, are equal to the sides CE, CF; and the bases DF, EF, are also equal: therefore the included angles at C, being equal (*d*), and adjacent, the straight line FC is drawn from the given point C, at right angles to AB (*e*); which was to be done.

Recite (*a*), prop. 3; (*b*), prop 1; (*c*), post. 1;
(*d*), prop. 8; (*e*), def. 10.

Cor. Hence two straight lines cannot have a common segment: for, if ABC, ABD, have the segment AB common, they cannot both be straight lines. Draw BE at right angles to AB (*a*): then if ABC be a straight line, EBC is a right angle; and if ABD be a straight line, EBD is a right angle; and so two right angles are unequal, which is impossible, (*b*): therefore the lines, ABC, ABD, which have the common segment AB, are not both straight lines.

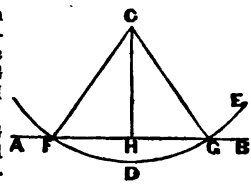



Q. E. D.

Recite (*a*), p. 11; (*b*), ax. 10.

12 P. To draw a straight line (CH), perpendicular to a given straight line (AB), of sufficient length, from a given point (C) on one side of it.

Construction. Take any point D, on the other side of AB; and from the centre C, at the distance CD, describe the arc EGF (*a*), cutting AB in F, G; bisect FG in H (*b*), and join CF, CH, CG, (*c*); CH is perpendicular to AB.

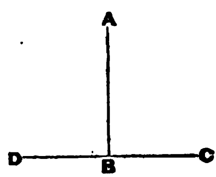



Argument. The triangles CHF, CHG have CH common, HF equal to HG, and the bases CF, CG are equal radii; therefore, the angles CHF, CHG, are equal (*d*), and being adjacent, they are right angles (*e*), and CH drawn from the point C, is consequently perpendicular to AB: which was to be done.

Recite (*a*) post. 3, (*b*) prop. 10, (*c*) post. 1, (*d*) prop. 8, (*e*) def. 10.

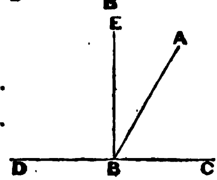
13 Th. The angles (ABC, ABD), made by one straight line (AB) with another (CD), upon one side of it, are either two right angles, or are equal to two right angles.

Construction. If the adjacent angles ABC, ABD, are equal to one another, they are right angles (*a*); if unequal, from the point B draw BE at right angles to CD (*b*).




Argument. The two right angles EBD, EBC are equal to the three EBD, EBA and ABC.

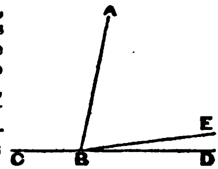
Again, the two angles ABD, ABC are equal to the three EBD, EBA, ABC; therefore ABC, ABD are equal to EBD, EBC (*c*), which were made equal to two right angles.




Therefore, the angles, &c. Q. E. D.

Recite (*a*), def. 10; (*b*), prop. 11; (*c*), ax. 1.

**14 Th. If, at a point (B), in a straight line (AB), two other straight lines (CB, DB) meet from opposite sides, making the adjacent angles equal to two right angles, these two lines shall be one continued straight line.**

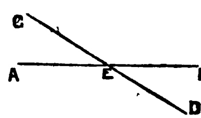



Argument. If CBD be not a straight line, make CBE such: therefore, since AB makes angles with the straight line CBE, on one side of it, the angles ABC, ABE, are equal to two right angles (*a*): but the angles ABC, ABD, are equal to two right angles. Take away the common angle ABC: therefore the remainders ABE, ABD, are equal (*b*); the less to the greater, which is impossible.

Wherefore, if at a point, &c. Q. E. D.

Recite (*a*), prop. 13; (*b*), ax. 3.

**15 Th. If two straight lines (AB, CD), cut one another, the opposite, or vertical angles (AEC, BED) and (BEC, AED) shall be equal to one another.**




Argument. Since the straight line AE makes with CD, the angles AEC, AED equal to two right angles; and the straight line DE makes with AB, the angles DEA, DEB equal to two right angles (*a*); from these equals take the common angle AED: therefore, the remainders AEC, BED are equal to one another (*b*).

In the same manner, it can be demonstrated that the angle AED is equal to the angle BEC.

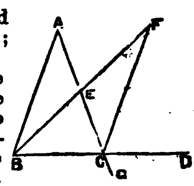
Therefore, if two straight lines, &c. Q. E. D.

Recite (*a*), prop. 13; (*b*), ax. 3.

Cor. 1. Hence if two straight lines cut one another, the angles made at the sectional point are equal to four right angles.

Cor. 2. Consequently, all the angles made about a point, by any number of cutting lines, are equal to four right angles.

**16 Th. If one side (BC) of a triangle (ABC) be produced, the exterior angle (ACD) is greater than either of the interior opposite angles (at A, or B).**




Constr. Bisect AC in E (*a*); join BE (*b*) and produce it to F (*c*); make EF equal to EB; join FC, and produce AC to G.

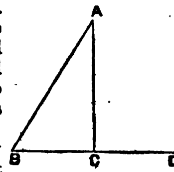
Argument. The triangles AEB, CEF are equal; having two sides EA, EB in the one equal to two sides EC, EF in the other; also their vertical angles at E being equal (*d*); therefore, their angles A and ECF are equal. But the exterior angle ECD is greater than ECF; or A.

In like manner, if BC be bisected, it may be proved that the angle BCG, or its equal ACD, is greater than the angle ABC.

Therefore, if one side of a triangle, &c. Q. E. D.

Recite (*a*), p. 10; (*b*), pos. 1;
(*c*), pos. 2; (*d*), p. 4, 15.

17 Th. Any two any angles of triangle (ABC) are together less than two right angles.



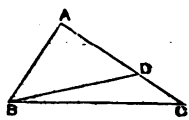

Argument. Produce the side BC to D (*a*). Then since the interior angle B is less than the exterior and opposite angle ACD (*b*), to each add ACB; then ACB and B are less than ACB and ACD (*c*): but these latter two are equal to two right angles (*d*); therefore, ACB and B are less than two right angles.

In the same way, it may be proved that any two of the angles of ABC are less than two right angles.

Q. E. D.

Recite (*a*), pos. 2; (*b*), p. 16; (*c*), ax. 4; (*d*), p. 13.

18 Th. In every triangle (ABC) the greater angle (B) is opposite to the greater side (AC).




Constr. Since the side AB is less than AC, cut off a part AD equal to AB (*a*); join BD (*b*).

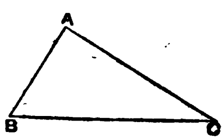
Argument. The exterior angle ADB, or its equal ABD (*c*), exceeds the interior C (*d*); much more does ABC exceed C.

Therefore, in every side, &c.

Q. E. D.

Recite (*a*), p. 3; (*b*), pos. 1;
(*c*), p. 5; (*d*), p. 16.

19 Th. The greater angle of every triangle is subtended by the greater side, and the less by the less.

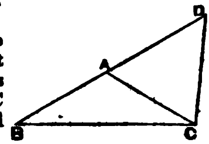



Argument. If in the triangle ABC, the angle B be greater than C, the side AC will exceed the side AB: for if not, it must be either equal to it, or less: equal it is not, because B is not equal to C (*a*); neither is it less, because B is not less than C. It remains, therefore, that AC is greater than AB, because it subtends a greater angle (*b*).

Q. E. D.

Recite (*a*), p. 5; (*b*), p. 18.

20 Th. Any one side of a triangle is less than the sum of the other two.



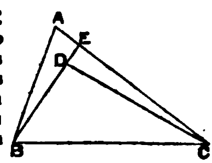
Constr. In the triangle ABC, produce BA so as to make AD equal to AC: join CD (*a*).

Argument. The angles ACD, ADC are equal, being opposite to equal sides (*a*); but either of them is less than BCD; and the less side subtends the less angle (*b*); therefore BC is less than BD, which is the sum of BA and AC.

In this manner it may be proved that either side is less than the sum of the other two. Q. E. D.

Recite (*a*), pos. 1; (*b*), p. 5, 19.

21 Th. If from a point (D) within a triangle (ABC), two straight lines (DB, DC) be drawn to the ends of the base; these lines shall be less than the two sides (AB, AC), but they shall contain a greater angle.



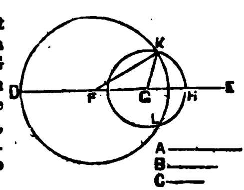
Argument 1. Produce BD to E (*b*). BE is less than the sum of BA, AE (*b*); add EC to both. Then the sum of BE and EC is less than the sum of BA and AC (*c*). Again CD is less than the sum of CE and ED (*b*); to both add DB; then the sum of CD, DB is less than the sum of CE, EB, and still less than the sum of CA, AB (*c*).

2. The angle BEC is exterior of the triangle BAE, and interior of the triangle CED; it is therefore less than the exterior BDC, and greater than the interior BAC: BDC is therefore greater than BAC (*d*).

Wherefore, if from a point, &c. Q. E. D.

Recite (*a*), pos. 2; (*b*), p. 20; (*c*), ax. 4; (*d*), p. 16.

22 P. To make a triangle of which the sides shall be equal to three given straight lines (A, B, C); but any one of them must be less than the sum of the other two.

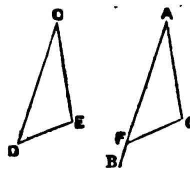


Constr. Take a straight line DE, not less than the sum of the three given straight lines; make DF equal to A, FG equal to B, and GH equal to C (*a*): from F as centre, and radius FD, describe the circle DKL (*b*); also from G, as centre, and radius GH, describe the circle KHL. The triangle FGK has its sides equal to the straight lines A, B, C.

Argument. FK, FD are equal radii (*c*); and FD equals A; therefore FK equals A (*d*): also GK, GH are equal radii; and GH equals C; therefore GK equals C. Therefore FK, FG, GK are equal to A, B, C, each to each; and the thing is done which was required.

Recite (*a*), p. 3; (*b*), pos. 3; (*c*), def. 15; (*d*), ax. 1.

23 P. At a given point (A) in a given straight line (AB) to make an angle equal to a given rectilineal angle (C).

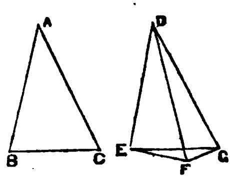



Constr. In the lines forming the angle C join any two points D, E (*a*); make the triangle AFG of sides equal to CD, CE, DE, each to each (*b*).

Argument. Since FA, AG are made equal to DC, CE, and FG to DE; therefore, at the point A, in the straight line AB, an angle is made equal to the angle C (*c*); which was to be done.

Recite (*a*), pos. 1; (*b*), p. 22; (*c*) p. 8.

24 Th. If two sides (AB, AC) of one triangle, be equal to two sides (DE, DF) of another, while the contained angles (A and D) are unequal; the base (EF) opposite the less angle (D) is less than the base (BC) opposite the greater angle (A).




Constr. Let DF be not less than DE: then, at the point D, in the straight line ED, make the angle EDG equal to A (*a*); make DG equal to AC, or DF (*b*); join EG, GF (*c*).

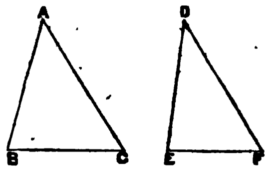
Dem. The two sides BA, AC being severally equal to the two ED, DG, and the angle A to EDG, the bases BC, EG are therefore equal (*d*). But, in the triangle EFG, EF is less than EG; for it subtends a less angle—that is, since DFG is isosceles (*d*), the angles DFG, DGF are equal; but EGF is less, and EFG is greater than either: therefore EF is less than BC.

Wherefore, if two sides, &c. Q. E. D.

Recite (*a*), p. 23; (*b*), cor. p. 3; (*c*), post. 1; (*d*), p. 5.

25 Th. If two sides of one triangle be severally equal to two sides of another, but the bases unequal; the angle opposite the greater base is greater than the other.

Dem. If, in the triangles ABC, DEF, the sides AB, AC be equal to the sides DE, DF, each to each,—but the base BC greater than the base EF; then the angle A is greater than the angle D. For if A be not greater than D, it must be equal to it, or less than it. Equal it is not, because BC is not equal to EF (*a*): neither is it less, because BC is not less than EF (*a*).



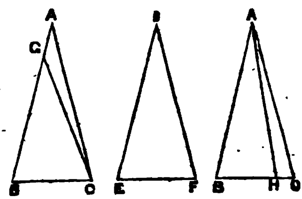

Wherefore, if two sides, &c. Q. E. D.

Recite (*a*), p. 24.

26 Th. Two triangles are equal to one another, in all their parts, which have two angles and a side in the one equal to two angles and a side in the other; whether the equal sides be adjacent to both, or only to one of the equal angles.

Let ABC, DEF be two triangles, having equal angles at B, E and at C, F.

1. Let BC=EF, which are adjacent to both the equal angles; then, the side AB, for example, will equal the side DE: but if not equal, let it be greater; so that a part of it, as BG, shall equal ED (*a*) join GC;—Hence the triangles BCG, EFD are equal, having the angles B, E equal, and the sides BC, BG in the one equal to the sides EF, ED in the other (*b*). Therefore, the angles BCG and F are equal; but the angles BCA and F are equal, by hypoth. therefore BCG a part, equals BCA the whole, which cannot be admitted. Therefore, AB is not greater than DE.



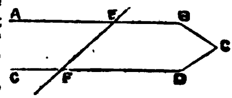

2. Let AB=DE, which are adjacent to the equals B, E, and opposite to C, F: then, the sides BC, EF, for example, are equal; but if not equal, let BC be the greater; so that a part of it, as BH, shall equal EF (*a*). Hence the triangles ABH, DEF are equal, having the angles B, E equal by hypoth. and the sides BA, BH equal to the sides ED, EF (*b*): therefore the angle BHA equals the angle F, or its equal C; that is, the exterior equals the interior on the same side (*c*), which is impossible. Therefore BC is not greater than EF.

Wherefore, two triangles are equal in all their parts, &c.

Q. E. D.

Recite (*a*), p. 3; (*b*), p. 4; (*c*), p. 16.

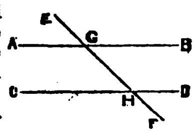
27 Th. If a straight line (EF) fall upon two other straight lines (AB, CD), making the alternate angles (AEF, EFD), equal to each other; these two straight lines shall be parallel.




Argument. For, if AB be not parallel to CD, produce them, and they shall diverge in one direction and meet in the other. Let them meet in the point G: therefore GEF is a triangle, whose exterior angle AEF exceeds its interior opposite angle EFG (*a*), which were said to be equal. Therefore AB and CD, neither meet nor diverge, by production, as was supposed; but are parallel, (*b*). Q. E. D.

Recite (*a*), p. 16; (*b*), Note def. 4.

28 Th. If a straight line (EF), falling upon other two straight lines (AB, CD), make equal the exterior angle (EGB) to the interior opposite angle (GHD on the same side of the line; or make the interior angles (BGH, GHD) equal to two right angles, the two lines shall be parallel.




Argument 1. Since, by hyp. EGB is equal to GHD, and also to AGH (*a*); therefore AGH is equal to GHD (*b*); and they are alternate angles (*c*); therefore AB is parallel to CD.

2. Since by hyp. BGH and GHD are equal to two right angles, and that AGH and BGH are also equal to two right angles (*d*), the former two are equal to the latter two (*b*); then taking out the common angle BGH, there remain AGH equal to GHD (*e*); and they are alternate angles (*c*): therefore AB is parallel to CD.

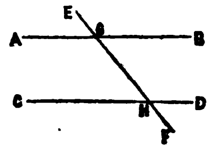
Wherefore, if a straight line, &c. Q. E. D.

Recite (*a*), p. 15; (*b*), ax. 1; (*c*), p. 27; (*d*), p. 13; (*e*), ax. 3.

29 Th. If a straight line (EF) cut two parallel straight lines (AB, CD), it makes equal 1, the alternate angles; 2, the interior and exterior angles on the same side; and 3, the two interior angles on the same side to two right angles.

1. The alternate angles AGH, GHD are equal, or not; if not, let AGH be the greater, and add BGH to each; then AGH and BGH exceed BGH and GHD (*a*): but AGH and BGH are equal to two right angles (*b*); therefore BGH and GHD are less than two right angles, and the lines AB, CD will meet towards B, D (*c*); but they are parallel and cannot meet; therefore the alternate angles AGH, GHD are equal, as stated.

2. The exterior and interior angles EGB, GHD, on the same side of EF are equal: for, as vertical angles EGB, AGH are equal (*d*); and AGH proves equal to GHD: therefore EGB equals GHD, as stated.



3. The interior angles BGH, GHD are equal to two right angles: for AGH and BGH are equal to two right angles (*b*); and AGH equals GHD: therefore BGH and GHD are equal to the same, as stated.

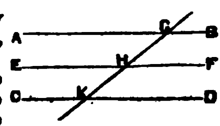
Wherefore, if a straight line, &c. Q. E. D.

Recite (*a*) ax. 4; (*b*) p. 13;
(*c*) Note def. 4; (*d*) p. 15.

**30 Th. Straight lines (AB, CD) which are parallel to the same straight line (EF), are parallel to each other.**

Let GHK cut the parallels AB, EF, also CD, EF; then AB is parallel to CD.

For since the parallels AB, EF are cut by a straight line, the alternate angles AGH, GHF are equal (*a*).



Also, because the parallels CD, EF are cut by a straight line, the exterior angle GHF equals the interior, opposite angle GKD (*b*).

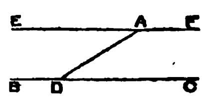
Therefore, each of the angles AGH, GKD, being equal to GHF, are equal to each other (*c*); and being alternate angles, AB is parallel to CD (*a*).

Wherefore, straight lines, &c. Q. E. D.

Recite (*a*) p. 27; (*b*) p. 28; (*c*) ax. 1.

**31 P. To draw a straight line through a given point (A), parallel to a given straight line (BC).**

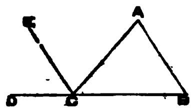
In BC take any point D, and join DA (*a*); make the angle DAE equal to the angle ADC (*b*); produce the straight line EA to F (*c*).



Because the alternate angles EAD and ADC are equal (*d*), the straight line EF is parallel to the given straight line BC; and it is drawn through the given point A, which was to be done.

Recite (*a*) pos. 1; (*b*) p. 23;
(*c*) pos. 2; (*d*) p. 27.

32 Th. If any side (BC) of a triangle be produced, the exterior angle (ACD) is equal to the two interior angles (A and B); and the three interior angles of every triangle are equal to two right angles.



Argument. Through C draw CE parallel to AB (*a*). Then because AC meets the parallels AB, CE, the alternate angles A and ACE are equal (*b*).

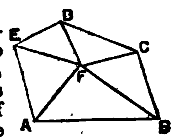
Also, because BD meets the parallels AB, CE, the exterior DCE equals the interior B (*c*).

Therefore, the interior angles A and B equal the angles ACE and DCE, that is, the exterior angle ACD.

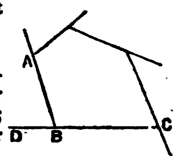
Again, to these equals add the angle ACB (*d*); therefore, the three interior angles A, B and ACB are equal to the two ACD, ACB, that is, to two right angles (*e*).

Wherefore, if any side, &c. Q. E. D.

Recite (*a*) p. 31; (*b*) p. 27; (*c*) p. 29; (*d*) ax. 2; (*e*) p. 13.

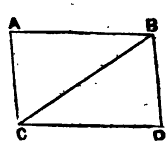


Cor. 1. All the interior angles of any rectilineal figure and four right angles, are equal to twice as many right angles as the figure has sides. For, about a point within the figure, as many triangles may be formed as the figure has sides, each of whose angles shall equal two right angles; and the angles about the point are equal to four right angles.



Cor. 2. All the exterior angles of any rectilineal figure are equal to four right angles; for such figure may be reduced to a mere point; about which there cannot be more than four right angles.

33 Th. The straight lines (AC, BD) which join the same ends of two equal and parallel straight lines (AB, CD, are themselves equal and parallel.



Join the alternate ends BC (*a*); then since the equal and parallel lines AB, CD are met by BC, the alternate angles ABC, BCD are equal (*b*); and because AB equals CD, and BC is common; therefore the bases AC, BD are equal (*c*); also the alternate angles ACB, CBD (*b*); therefore also AC is parallel to BD.

Wherefore, the straight lines which join, &c.

Q. E. D.

Recite (*a*) pos 1; (*b*) p. 27; (*c*) p. 4.

**Definition.** **A parallelogram is a quadrilateral figure whose opposite sides are parallel.**

34 Th. The opposite sides and angles of a parallelogram are equal, and the diameter bisects it (*a*).

Because ABCD is a parallelogram, its opposite sides are parallel (*b*); and because BC joins opposite angles (*c*) it meets the parallels AB, CD, and also the parallels AC, BD, and makes the two angles at B alternately equal to the two at C (*d*); so that the whole angles ABD, ACD are equal.

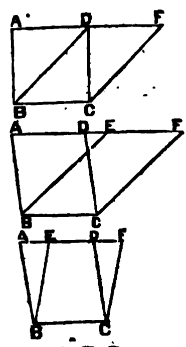
Again, in the triangles ABC, DCB, the side BC is common; and their angles at B and C are alternately equal; therefore the remaining angles at A and D are equal (*e*); also the sides AB to CD and AC to BD (*f*); therefore each of the triangles ABC, BCD is half of the parallelogram, which is bisected by BC.

Wherefore, the opposite sides and angles, &c., Q. E. D.

Recite (*a*), p. 9, 10; (*b*), def. above; (*c*), def. 16; (*d*), p. 27; (*e*), p. 32; (*f*), p. 26.

35 Th. Parallelograms upon the same base (BC), and between the same parallels (AF, BC), are equal to one another.

To make ABCD, BCFD, distinct parallelograms on the same base, AF must be greater than BC; and the figure ABCF will be a trapezoid (*a*).



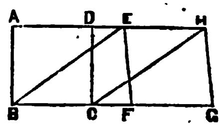
1. If AF be the double of BC, the straight lines DB, DC will bisect AF, and also each of the parallelograms; and the triangle DBC will be the half of each (*b*); hence the parallelograms ABCD, BCFD will be equal (*c*).

2. But if AF be greater than twice BC, DE will be the excess; and if less, DE will be a common segment of AD, EF. To each, therefore, add the excess, or from each subtract the segment; the sums, or differences, AE, DF, will be equal (*d*). In either case, therefore, the sides CD, DF equal the sides BA, AE; and the exterior angle CDF equals the interior and opposite angle BAE (*e*), and the triangles CDF, BAE are equal (*f*). From the trapezoid ABCF take each of these triangles; the remaining parallelograms ABCD, BCFE are equal (*d*).

Wherefore, parallelograms, &c. Q. E. D.

Recite (*a*), def. 34; (*b*), p. 34; (*c*), ax. 6; (*d*), ax. 2, 3; (*e*), p. 32; (*f*), p. 4.

**36 Th. Parallelograms upon equal bases, and between the same parallels, are equal to one another.**



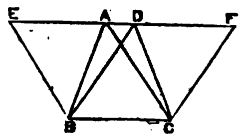

Let ABCD, EFGH be two parallelograms upon equal bases BC, FG, and between the same parallels AH, BG; they are equal.

Join BE, CH; and because the bases BC, FG are equal, and FG is equal to EH (*a*); therefore BC and EH are equal, and they are parallel; therefore also BE and CH, which join their extremities, are equal and parallel (*b*). EBCH is therefore a parallelogram, and equal to ABCD; because they are upon the same base BC, and between the same parallels BC, AH (*c*). For the like reason, EFGH is equal to EBCH; hence ABCD equals EFGH (*d*).

Wherefore, parallelograms, &c. Q. E. D.

Recite (*a*) p. 34; (*b*) p. 33; (*c*) p. 35; (*d*) ax. 1.

**37 Th. Triangles (ABC, DBC), upon the same base (BC), and between the same parallels (BC, AD) are equal to one another.**



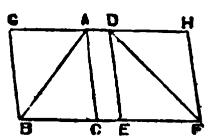

Produce AD both ways to E, F; through B draw BE parallel to AC (*a*); and through C draw CF parallel to BD.

Each of the figures EBCA, DBCF is a parallelogram; and they are equal (*b*), because they are upon the same base and between the same parallels, BC, EF: and the triangle ABC is half the parallelogram EBCA, because the diameter AB bisects it (*c*); and the triangle DBC is half the parallelogram DBCF, because the diameter DC bisects it; and the halves of equals are equal (*d*). Therefore, the triangle ABC is equal to the triangle DBC.

Wherefore, triangles, &c. Q. E. D.

Recite (*a*) p. 31; (*b*) p. 35; (*c*) p. 34; (*d*) ax. 7.

**38 Th. Triangles (ABC, DEF), upon equal bases (BC, EF), and between the same parallels (BF, AD), are equal to one another.**




Produce AD both ways to G, H; and through B, draw BG parallel to AC (*a*); and through F, draw FH parallel to ED.

Then each of the figures BCAG, FEDH is a parallelogram; and they are equals (*b*), being upon equal bases and between the same parallels BF, GH.

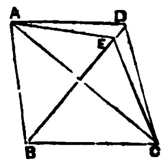
But the triangle DEF is half the parallelogram DEFH; and the triangle ABC is half the parallelogram GBCA

(*c*): because they are bisected by the diameters DF and AB: but the halves of equals (*d*) are equal; therefore the triangles ABC and DEF are equal.

Wherefore, triangles, &c. Q. E. D.

Recite (*a*) p. 31; (*b*) p. 36; (*c*) p. 34; (*d*) ax. 7.

39 Th. Equal triangles (ABC, DBC), upon the same base (BC), and upon the same side of it, are between the same parallels.




Join AD;—AD is parallel to BC: for if not, through the point A draw AE parallel to BC (*a*), and join CE.

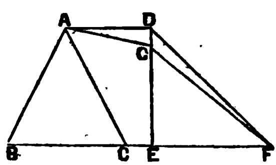
The triangles ABC, EBC, are equal (*b*), because they are on the same base BC, and between the same parallels BC, AE: but the triangles DBC and ABC are equal by hyp., therefore the triangles EBC and DBC are equal, the less to the greater, which is absurd.

Therefore AE is not parallel to BC; neither is any other line but AD parallel to BC.

Wherefore, equal triangles, &c. Q. E. D.

Recite (*a*) p. 31; (*b*), p. 37.

40 Th. Equal triangles (ABC, DEF), upon equal bases (BC, EF), in the same straight line (BF), on one side of it, are between the same parallels.



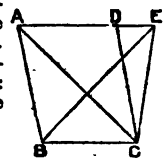

Join AD;—AD is parallel to BC; for if not, through A (*a*) draw AG parallel to BF, and join GF. The triangle ABC is equal to the triangle GEF, because they are upon equal bases BC, EF, and between the same parallels BF, AG (*b*).

But the triangle ABC is equal to the triangle DEF; therefore GEF and DEF are equal, the less to the greater, which is impossible. Therefore AG is not parallel to BF; neither is any other line but AD.

Wherefore, equal triangles, &c. Q. E. D.

Recite (*a*) p. 31; (*b*) p. 38.

41 Th. If a parallelogram (ABCD), and a triangle (EBC), be upon the same base BC, and between the same parallels (BC, AE), the parallelogram is double of the triangle.



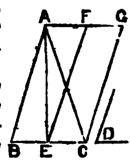

Join AC;—then the triangle ABC equals the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (*a*). But the parallelogram ABCD is double of the triangle ABC (*b*), because the diameter bisects it: wherefore ABCD is also double of the triangle EBC (*c*).

Therefore, if a parallelogram, &c.

Q. E. D

Recite (*a*) p. 37; (*b*) p. 34; (*c*) ax. 7.

42 P. To describe a parallelogram that shall be equal to a given triangle (ABC,) and have one of its angles equal to a given rectilineal angle (D).

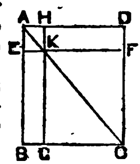



Bisect BC in E (*a*); join AE, and at the point E, in the straight line EC, make the angle CEF equal to D (*b*); and through A, draw AG parallel to EC; and through C draw CG parallel to EF (*c*).

EFGC is therefore a parallelogram; and because BE and EC are equal, the triangles AEB, AEC are also equal, since they are upon equal bases, and between the same parallels BC, AG (*d*); therefore, the triangle ABC is double of the triangle AEC: the parallelogram CEFG is likewise double of the same triangle, being upon the same base and between the same parallels (*e*); CEFG is therefore equal to ABC, and has an angle equal to the angle D. Therefore the thing required has been done.

Recite (*a*) p. 10; (*b*) p. 23; (*c*) p. 31;
(*d*) p. 38; (*e*) p. 41.

43 Th. The complements of the parallelograms about the diameter (AC) of any parallelogram (ABCD), are equal to one another.




The parallelograms about the diameter AC, or through which AC passes, are EH, GF; and the complements, which make up the whole figure, are BK, KD, which are said to be equal.

The diameter AC bisects the parallelogram ABCD; its parts AK, KC bisect also EH, GF (*a*); therefore, the triangles ABC, ADC are equal; also the triangles AEK, AHK, and KGC, KFC.

Therefore, from ABC take AEK+KGC, the remainder is BK.

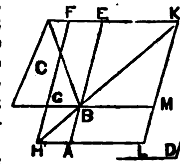
Also, from ADC take AHK+KFC, the remainder is KD.

But taking equals from equals the remainders are equal (*b*); therefore BK is equal to KD.

Wherefore, the complements, &c. Q. E. D.

Recite (*a*) p. 34; (*b*) ax. 3.

**44 P. To a given straight line (AB) to apply a parallelogram equal to a given triangle (C), having an angle equal to a given rectilineal angle.**



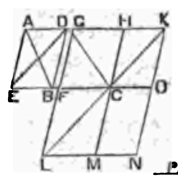

Upon half the base of the given triangle C, make a parallelogram GBEF, equal to C (*a*); and let its angle GBE equal the given angle D (*b*); let AB, the given straight line, be the produced part of EB; through B produce GB to M; through A draw HL parallel to GM; produce FG to H; through B draw HK; produce FE to K; make KL parallel to EA.

The complements BL and BF are equal (*c*); but BF was made equal to the given triangle C; therefore, BL is equal to C (*d*); and it is applied to the given straight line AB, having one of its angles ABM, equal to the opposite vertical angle GBE, which was made equal to the given angle D.

Therefore the thing required has been done.

Recite (*a*) p. 42; (*b*) p. 23; (*c*) p. 43; (*d*) ax. 1.

**45 P. To describe a parallelogram equal to a given rectilineal figure (ABCD), having an angle equal to a given rectilineal angle (P); and to apply the parallelogram to a given straight line (CM).**




Produce the side CB of the given rectilineal figure indefinitely to E (*a*); join DB (*b*), and parallel to it, through A, draw AE (*c*); join DE (*b*).

Again, bisect CE in F (*d*); upon CF describe the parallelogram CFGH, equal to the triangle DEC, and having an angle FCH equal to the given angle P (*e*).

Also, let CM, the line to which the parallelogram is to be applied, be in the same straight line with HC; produce FC to O; through M draw LN parallel to FO (*c*); produce GF to meet NL; through C draw LK; produce GH to meet LK; parallel to HCM draw KON (*c*.)

1. The triangles DAB, DEB, on the same base DB, and between the same parallels DB, AE, are equal to one another (*f*); to each add the triangle DBC; therefore, the triangle DEC is equal to the given rectilineal figure ABCD (*g*).

2. But the parallelogram CFGH was made equal to the triangle DEC; therefore CFGH is equal to the given rectilineal figure ABCD (*h*), and its angle FCH is equal to the given angle P.

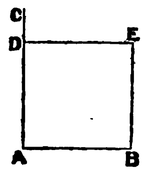
3. The complements FH and MO are equal (*i*); and the angle MCO is equal to its vertical and opposite angle FCH, and therefore to the given angle P (*k*). Wherefore to CM is applied a parallelo-

gram equal to a given rectilineal figure, having an angle equal to a given rectilineal angle.

Recite (*a*) pos. 2; (*b*) pos. 1; (*c*) p. 31; (*d*) p. 10; (*e*) p. 42; (*f*) p. 37; (*g*) ax. 4; (*h*) ax. 1; (*i*) p. 43; (*k*) p. 15;

NOTE.—This demonstration includes the corollary, as given in Simson's and Playfair's edition.

## 46 P. To describe a square upon a given straight line (AB.)




Constr. From the point A draw AC at right angles to AB (*a*); make AD equal to AB (*b*); draw DE parallel to AB, and BE parallel to AD (*c*).

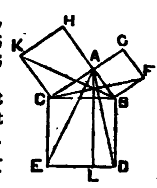
1. Of the equal angles. The angle A is a right angle; and, because of the parallels, the two interior angles, A and D, are equal to two right angles (*d*); also, for the same reason, A equals B, and B equals E. Therefore, the figure has four right angles.

2. Of the equal sides. The sides AB, AD are made equal; and because the opposite sides of a parallelogram are equal (*e*), DE is equal to AB, and BE to AD. Hence also, the figure has four equal sides; and is therefore a square (*f*).

Wherefore, the required square has been described.

Recite (*a*) p. 11; (*b*) p. 3; (*c*) p. 31; (*d*) p. 29; (*e*) p. 34; (*f*) def. 30.

## 47 Th. In any right-angled triangle (ABC), the square upon the side (BC), subtending the right angle, is equal to the sum of the squares of the sides (AB, AC), containing the right angle.




Construction. On BC describe the square BE, on AB the square BG, on AC the square AK (*a*); draw AL parallel to BD, or CE (*b*); join AD, CF; also AE, BK.

Dem. Because of the adjacent right angles at A, the lines BA, AH, and CA, AG, are straight lines (*c*), the *one* parallel to BF, the *other* to CK.

The triangle BCF is upon the side BF, and between the same parallels as the square BG, of which it is equal to the half:

Also, the triangle ABD is upon the side BA, and between the same parallels as the parallelogram BL, of which it is equal to the half (*d*).

But the triangles BCF, ABD are equal; having two sides BF, BC, in the *one*, equal to two sides, BA, BD, in the *other;* and each of the two containing the angle ABC and a right angle (*e*).

Therefore, the half of the square BG, is equal to the half of the parallelogram BL; hence the square is equal to the parallelogram (*f*).

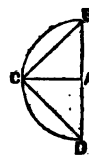
In the same way, it may be shown that the square CH, is equal to the parallelogram CL.

Therefore, the two parallelograms BL, CL, viz. the square BE, are equal to the two squares, BG, CH.

Wherefore, the square upon BC is equal to the sum of the squares upon AB, AC. Q. E. D.

Recite (*a*) p. 46; (*b*) p. 31; (*c*) p. 14;
(*d*) ax. 7; (*e*) ax. 2; (*f*) ax. 6 and p. 41.

**48 Th. If the square upon one side (BC) of a triangle be equal to the squares upon the other two sides (AB, AC), the angle contained by these two sides shall be a right angle.**




From the point A, draw AD at right angles to AC (*a*); and make AD equal to AB; join CD.

Then because AD equals AB, their squares also are equal;—to each of them add the square of AC; therefore the squares of AD, AC equal the squares of AB, AC (*b*).

But the square of CD equals the squares of AD, AC (*c*), because DAC is a right angle: and the square of BC was supposed to be equal to the squares of AB, AC. Therefore, the squares of BC and CD are equal, and BC is equal to CD.

And because the sides AD, AB are equal, and AC common to the two triangles, and the bases BC, CD also equal; therefore the angle BAC is equal to the angle DAC (*d*): but DAC is a right angle; therefore BAC is also a right angle.

Wherefore, if the square, &c. Q. E. D.

Recite (*a*) p. 11; (*b*) ax. 2;
(*c*) p. 47; (*d*) p. 8—all of b. 1.

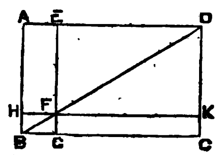
# BOOK SECOND.

## *Definitions.*

THE multiplication or division of magnitudes makes no change in their species; and magnitudes of different kinds cannot be united in additions: a part taken from a magnitude is of the same species as the whole. The units of a line are therefore lines;—of a superficies, areas;—of a solid, solids;—of an angle, angles.

Any produced magnitude is therefore a multiple of its root or basis; or of the unit which measures the basis.

1. A rectangle is a superficies contained under two straight lines at right angles to each other: if the lines are equal, the figure is a square; if unequal, the figure is an oblong; and in either case, it is a right angled parallelogram.




2. A gnomon is the sum of the two complements (p. 43,) and one of the parallelograms about the diameter of a rectangle.

NOTE 1. The operative signs +, —, ×, ÷, =, are sometimes used to express the sum, difference, product, quotient and equality of magnitudes; and the parentheses () unite two or more magnitudes in one. A point is often used to denote multiplication instead of the oblique cross.

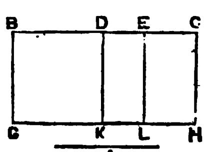
NOTE 2. Because a square has two equal dimensions, the figure 2 placed over one of them saves repetition; thus, $AB^2$ is the square of AB; but in the case of a rectangle of different dimensions, both must be expressed; thus, AB×BC, or AB.BC, is the rectangle or product of AB multiplied into BC.

NOTE 3. Lines used numerically, as multiplier and multiplicand, are called coefficients of each other, or co-factors.

## *Propositions.*

1 Th. The rectangle contained under two straight lines (A and BC), is equal to the several rectangles contained under one of them and all the parts of the other.

Let A be a whole line and BC a divided one, as in the points D, E.




Construction. Draw BG at right angles to BC *(a)*, and make it equal to A *(b)*; draw GH parallel to BC *(c)*, and DK, EL, CH parallel to BG. The opposite parallels are equal *(d)*.

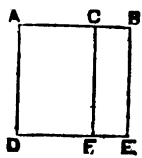
Argument. Now it is obvious that the

rectangle BH contains the three rectangles BK, DL, EH; and BH is the rectangle of BG and BC: and, since BG, DK, EL are each of them equal to A (*d*), therefore A×BD equals BK, A×DE equals DL, and A×EC equals EH; but BD, DE, EC are all the parts of BC: therefore, the rectangle of A and BC is equal to the rectangles of A and all the parts of BC.

Wherefore, the rectangle contained, &c. Q. E. D.

Recite (*a*) p. 11; (*b*) p. 3;
(*c*) p 31; (*d*) p. 34—all of book 1.

**2 Th. The square of a straight line is equal to two rectangles contained under that line and any two parts into which it may be divided.**



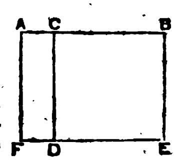

Let AB be a straight line divided into the parts AC, CB. Describe on AB the square AE (*a*); draw CF parallel to AD (*b*).

Now the square AE equals the two rectangles AF and CE: but AE is the square of AB (*c*); and AF is the rectangle of AD and AC, and CE is the rectangle of CF and CB. Also, since AE is a square, AD, or CF is equal to AB, and AC, CB are the parts.

Therefore, the square of a straight line, &c. Q. E. D.

Recite (*a*) p. 46; (*b*) p. 31—all of b. 1;
(*c*) def. 30, b. 1, and 1 of b. 2.

**3 Th. The rectangle contained by a straight line (AB) and (BC) one of two parts into which it is divided, is equal to the square of that part and the rectangle of the two parts.**



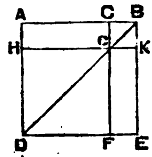

Constr. Upon BC describe the square BD (*a*); produce ED F (*b*) to meet AF drawn parallel to CD, or BE (*c*).

Then, since BE and BC, sides of a square, are equal, AE is the rectangle of AB and BC; and it contains the square of BC with the rectangle of AC and CD, or CB, and is equal to them.

Wherefore, the rectangle contained, &c. Q. E. D.

Recite (*a*) p. 46; (*b*) pos. 2; (*c*) p. 31—all of b. 1.

4 Th. The square of a straight line (AB) is equal to the squares of any two parts (AC, CB) into which it may be divided, and two rectangles of the parts




Constr. Upon AB describe the square AE (*a*); join BD; draw CF parallel to AD, meeting BD in G; draw HGK parallel to AB (*b*).

Argument. The angles CGB, ADB, as exterior and interior, are equal (*c*); and ADB equals ABD, from the isosceles (*d*); therefore, ABD equals CGB (*e*), and the sides CG, CB are equal (*f*), and their opposites BK, GK are equal (*g*). CK is therefore a square (*h*) on CB.

For similar reasons, HF is a square on HG, which is equal to AC. It is also plain that AG is the rectangle of AC and CG, or CB; also, that GE is the rectangle of GF and GK, which are equal to AC and CB.

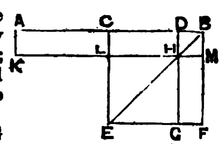
Now the two squares and two rectangles make up the square of AB. Wherefore, the square, &c. Q. E. D.

Recite (*a*) p. 46; (*b*) p. 31; (*c*) p. 29; (*d*) p. 5; (*e*) ax. 1; (*f*) p. 6; (*g*) p. 34—all of b. 1: (*h*) def. 30 of b. 1, and 1 of b. 2.

Cor. The rectangles about the diameter of a square are squares.

Scholium. A line or other magnitude divided into two parts, is called a binomial, the properties of which are very remarkable.

5 Th. If a straight line (AB), be divided into two equal parts (in C), and two unequal parts (in D), the square of half the line is equal to the rectangle of the unequal parts and the square of the line between the sectional points.




Upon CB, half the line, describe the square CF (*a*); join BE; through D, draw DG parallel to BF (*b*), meeting BE in H; through H, draw ML parallel to AB, and produce it to meet AK drawn parallel to CE.

The square CF equals the gnomon CMG and the square LG. But the gnomon is equal to the rectangle AH; for the complements CH, HF are equal (*c*); and with DM added to each, CM equals DF; but CM equals AL, (*d*); therefore DF equals AL, (*e*).

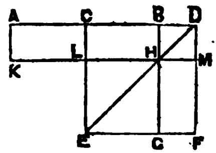
Again, AH is the rectangle of the unequal parts AD, DB; for DH equals DB (*f*): LG is also the square of LH, or CD, the line between the sectional points.

Therefore, the square of CB, is equal to the rectangle of AD, DB and the square of CD.

Wherefore, if a straight line, &c. Q. E. D.

Recite (*a*) p. 46; (*b*) p. 31; (*c*) p. 43; (*d*) p. 36; (*e*) ax. 1—all of b. 1; (*f*) cor. p. 4 of b. 2.

6 Th. If a straight line (AB) be bisected (in C), and produced to any point (D); the square of the line (CD), composed of the half and the part produced, is equal to the rectangle of the whole line AD and its produced part, (BD), with the square of (CB), half the line bisected.




Upon CD describe the square CF (*a*); join DE; through B, draw BG parallel to DF, and meeting the diameter in H; through H, draw ML parallel to AB, and produce it to meet AK drawn parallel to CE (*b*).

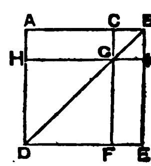
The square CF is equal to the gnomon CMG and the square LG: this is evident. But the gnomon is equal to the rectangle AM: for the rectangles AL and CH are equal (*c*); also CH and HF (*d*); therefore AL equals HF (*e*).

Hence the square CF equals the rectangle AM and the square LG. But AM is the rectangle of the whole line AD and its produced part BD: for DM or BH equals BD (*f*). Also, LG is the square of LH, or CB, half the bisected line.

Wherefore, the square of CD equals the square of CB and the rectangle of AD and DB. Q. E. D.

Recite (*a*) p. 46; (*b*) p. 31; (*c*) p. 36; (*d*) p. 43; (*e*) ax. 1; (*f*) p. 6, all of b. 1, and cor. p. 4, of b. 2.

7 Th. The two squares described upon a straight line (AB) and a part of that line (BC) are equal to two rectangles of the line and that part, together with the square of the other part (AC).




Constr. Upon AB describe the square AE (*a*) join BD; through C draw CF parallel to BE, and cutting the diameter in G; through G draw HK parallel to AB (*b*).

The squares AE, CK are equal to the rectangles AK, CE and the square HF. This is evident.

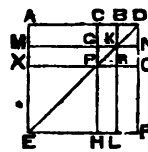
Now AE is the square of AB, and CK is the square of BC. Also, AK is the rectangle of AB and BK, or BC; and CE is the rectangle of BC and BE, or BA; and HF is the square of HG, or AC.

Wherefore, the two squares described, &c. Q. E. D.

Recite (*a*) p. 46 of b. 1; (*b*) p. 31 of b. 1.

Cor. Hence the sum of the squares of any two lines, is equal to twice the rectangle of the two with the square of their difference.

**8 Th. If a straight line (AB), be divided into any two parts (in C), four rectangles of the whole line and *one* part (CB), together with the square of the other part (AC), are equal to the square of the line (AD), composed of the whole and the *one* part.**



Produce AB, so that BD be equal to CB; upon AD describe the square AF (*a*); join DE; draw CH, BL, parallel to DF, cutting the diameter in K, P; through K, P, draw MN, XO, parallel to AD (*b*).

The complements AK, KF are equal; as also the complements MP, PL (*c*); to these latter equals add the equals GR, BN—MR is therefore equal to PL and BN; and AK, KF, MR, PL and BN—namely, the gnomon AOH (*d*), are equal to four rectangles of AB and BC: to this gnomon add XH, which is the square of XP, or AC. All these, therefore, coincide, and fill the same space, with the figure AEFD, which is the square of AD, composed of AB and BC.

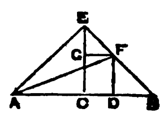
Wherefore, if a straight line be divided, &c. Q. E. D.

Recite (*a*) p. 46, 1; (*b*) p. 31, 1;
(*c*) p. 43, 1; (*d*) def. 2, 2.

Cor. 1. Four rectangles of any two lines, together with the square of their difference, are equal to the square of their sum; for, in this case, AD is the sum and AC the difference of AB, BC

Cor. 2. The square of any line is equal to four times the square of its half.

**9 Th. If a straight line (AB) be equally divided (in C) and unequally (in D); the squares of the two unequal parts (AD, DB), are double of the squares of half the line (AC) and of the line (CD) between the points of section.**



Constr. From C draw CE at right angles to AB (*a*) and equal to CA, or CB; join EA, EB; draw DF parallel to CE (*b*); join AF, and draw FG parallel to AB (*b*).

Argument. The sides CE, CA, CB being made equal, their opposite angles are equal (*c*); and on account of the right angles at C, the two angles at E make one right angle (*d*).

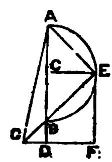
Again, the parallels CE, DF make the exterior angle DFB equal to the interior CEF, or its equal B (*e*): wherefore DF equals DB (*f*). Hence the side AF subtends two right angles, one at D and the other at E; and the square of AF equals the squares of the sides containing each of those angles (*g*): therefore the squares of AE, EF are equal to those of ADD, F, or AD, DB (*h*). But the square of AE equals the squares of AC, CE, or twice that of AC; and the square

of EF equals the squares of GE, GF, or twice that of GF, or its equal CD. Wherefore also, the squares of AD, DB are double the squares of AC and CD.

Therefore, if a straight line, &c. Q. E. D

Recite (*a*) p. 11; (*b*) p. 31; (*c*) p. 5;
(*d*) p. 32; (*e*) p. 29; (*f*) p. 6;
(*g*) p. 47; (*h*) ax. 1—all of b. 1.

10 Th. If a straight line (AB) be bisected (in C), and produced to any point (D); the square of this whole line (AD), and the square of its produced part (BD), are equal to two squares of (AC) half the bisected line, and two squares of (CD) the line composed of the half and the production.




From C draw CE at right angles to AB (*a*) and equal to CA, or CB; make EF equal and parallel to CD—then will FD be equal and parallel to CE (*b*); join EA, EB, and producing EB, FD, they will meet in G, because the angles F and GEF are less than two right angles C; join also AG.

Because the triangles ACE, BCE are right angled, the right angle is equal to half the sum of the angles in each (*d*), and because they are isosceles the angles opposite to equal sides are equal to each other (*e*), and each is half a right angle; therefore, AEB is a right angle.

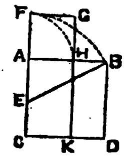
Also, since CBE is half a right angle, the alternate angle FEG is the same (*f*), and so is also the vertical opposite angle DBG (*g*). Again, because ECD is a right angle, the alternate angle CDG is the same: therefore DGB is half a right angle; and the sides DG and DB are equal; also the sides FG and FE (*h*).

AG therefore subtends two right angles; namely, AEG, ADG; and its square is equal to that of any two sides containing those angles (*i*). Wherefore, the squares of AE and EG are equal to the squares of AD and DG=BD, each pair being equal to the square of AG (*k*).

But the square of AE equals the squares of AC, CE, or two squares of AC; and the square of EG equals the squares EF, FG, or two squares of EF=CD. Therefore, the squares of AD, BD are equal to two squares of AC and two of CD. Q. E. D.

Recite (*a*) p. 11; (*b*) p. 31, 34; (*c*) p. 29;
(*d*) p. 32; (*e*) p. 5; (*f*) p. 27;
(*g*) p. 15; (*h*) p. 6; (*i*) p. 47;
(*k*) ax. 1—all of b. 1.

11 P. To divide a given straight line (AB) into two parts (in H), so that the rectangle of the whole line and one of the parts (HB) shall equal the square of the other part (AH).



Constr. Upon AB describe the square AD ($a$) bisect AC in E ($b$); join EB, and produce EA so that EF shall equal EB ($c$); upon AF describe the square AG ($a$), and produce GH to K.

Argument. The straight line AC being bisected in E and produced to F, the rectangle CF×FA and the square of AE are together equal to the square of EF ($d$), or its equal EB, or the squares of AE and AB ($e$). Take the square of AE from both; then the rectangle CF×FA—that is, FK, is equal to the square of AB, viz. AD ($f$). Take also from the rectangle and square the part AK, which is common; the remainders are equal; namely, AG to HD ($f$). But AG is the square of AH; and HD is the rectangle of HB×BD, or HB×AB. The straight line AB is therefore divided as required.

Q. E. D.

Recite ($a$) p. 46; ($b$) p. 10; ($c$) p. 3 of b. 1;
($d$) p. 6, of b. 2; ($e$) p. 47;
($f$) ax. 3 of b. 1.

12 Th. In any obtuse angled triangle, if a perpendicular be drawn from one of the acute angles to the opposite side produced; the square of the side subtending the obtuse angle equals the two squares of the sides containing it, and two rectangles of the produced side and its production.

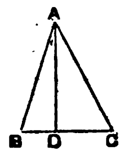
Let the triangle ABC be obtuse angled at C; produce BC to meet a perpendicular from A, in the point D. The production is CD, and AB subtends the obtuse angle.

Now $AB^2 = AD^2 + BD^2$ ($a$).
And $AD^2 = AC^2 - CD^2$ ($a$).
Also $BD^2 = BC^2 + CD^2 + 2BC \times CD$ ($b$).
Therefore, omitting $CD^2$, which is both positive and negative, $AB^2 = AC^2 + BC^2 + 2BC \times CD$ ($c$).
Wherefore, in any obtuse angled triangle, &c. Q. E. D.

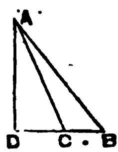
Recite ($a$) p. 47 of b. 1; ($b$) p. 4 of b. 2;
($c$) ax. 1.

13 Th. In any triangle (ABC), the square of the side (AC) subtending one of the acute angles (B) is less than the squares of the two containing sides (AB, BC), by two rectangles of one of them (BC) and a segment of the same intercepted between the said angular point and a perpendicular (AD) drawn to that side from the opposite angle (A).

1. Let AD meet BC within the triangle.
Then $BC^2+BD^2=2BC\times BD+CD^2$ (*a*):
And $AB^2 \quad =AD^2+BD^2$(*b*):
Add these two equations, omitting $BD^2$ from each side.
Then $AB^2+BC^2=AD^2+CD^2+2BC\times BD$: (*c*)
But $AC^2 \quad =AD^2+CD^2$. Now take this from the last equation: there remains
$AB^2+BC^2-AC^2=2BC\times BD$.



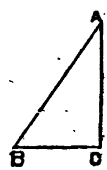

2. Let AD meet BC produced: then ACB is an obtuse angle; therefore $AB^2=AC^2+BC^2+2BC\times CD$ (*c*). To both sides add $BC^2$: then $AB^2+BC^2=AC^2+2BC^2+2BC\times CD$ (*d*). But since BD is divided in C, $BC^2+BC\times CD=BC\times BD$ (*e*); and the doubles are equal; therefore $2BC^2+2BC\times CD=2BC\times BD$ (*f*). Now substitute this in the equation above: it makes $AB^2+BC^2=AC^2+2BC\times BD$.




3. If AC be itself the perpendicular on BC; then BC is intercepted between the angular point B and the foot of AC; and $AB^2+BC^2=AC^2+2BC\times BC$ (*b*).

Wherefore, in every triangle, &c. Q. E. D.

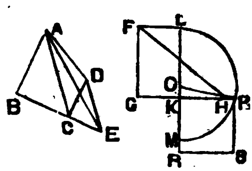
Recite (*a*) p. 7, 2; (*b*) p. 47, 1;
(*c*) p. 12, 2; (*d*) ax. 2;
(*e*) p. 3, 2; (*f*) ax. 6.




## 14 P. To describe a square that shall be equal to the given rectilineal figure (ABCD).

Constr. 1. Join the points A, C, of the given figure; and parallel to AC draw DE (*a*) to meet the base BC produced in E; join AE.

Argument 1. The triangles ACD, ACE upon the same base AC, and between the parallels AC, DE are equal (*b*): to each add the triangle ABC: then the quadrilateral ABCD equals the trilateral ABE (*c*).



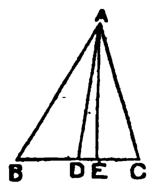

Constr. 2. Transfer the base BE to GH: place the triangle FGH at the same parallel distance as that of ABE; make FGH a right angle (*d*); upon GK, half the base GH, describe the rectangle GL (*d*): then, if the sides of GL are equal, the thing is done; for it is a square, and equal to the given figure. But if the sides are unequal, produce LK so that KM shall equal KG (*e*); bisect LM in O (*f*), at which centre describe the arc LPM (*g*); produce GH to meet the circle in P; join OP, and upon KP describe the square KS (*h*).

Argument 2. The straight line LM being divided equally in O and unequally in K, the rectangle $LK\times KM$, with the square of OK, is equal to the square of OM, or OP (*i*), or to the squares of OK, KP (*k*). From these equals reject the square of OK, the rectangle $LK\times$

KM is equal to the square of KP ($l$). But the rectangle LK, KM= KG, equals the triangle FGH, or ABE, or the figure ABCD. Therefore the square of KP, that is, KS, is equal to the same figure; and so, the thing is done which was required.

Recite ($a$) p. 31; ($b$) p. 37; ($c$) ax. 2; ($d$) p. 11, 42; ($e$) pos. 2; ($f$) p. 10; ($g$) pos. 3; ($h$) p. 46—all of b. 1; ($i$) p. 5, of b. 2; ($k$) p. 47, 1; ($l$) ax. 3.

A Th. If one side (BC) of a triangle be bisected (in D), the sum of the squares of the other two sides (AB, AC), is double of the square of that side, and of the square of the line (AD) drawn from the bisectional point to the opposite angle.




Draw AE at right angles to BC ($a$); then $AB^2 = AE^2 + BE^2$, and $AC^2 = AE^2 + CE^2$ ($b$);—adding these equals, therefore, $AB^2 + AC^2 = 2AE^2 + BE^2 + CE^2$ ($c$).

But since the straight line BC is cut equally in D, and unequally in E, the squares of BE and CE are equal to double squares of BD and DE ($d$).

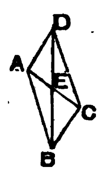
Therefore $AB^2 + AC^2 = 2BD^2 + 2DE^2 + 2AE^2$.

Now, the squares of AE, DE equal the square of AD ($e$). Therefore $AB^2 + AC^2 = 2BD^2 + 2AD^2$.

Q. E. D.

Recite ($a$) p. 12, 1; ($b$) p. 47, 1; ($c$) ax. 2; ($d$) p. 9, 2; ($e$) p. 47, 1.

B Th. In any parallelogram (ABCD), the sum of the squares of the diameters (AC, BD) is equal to the sum of the squares of its sides, (AB, AD, CB, CD.)




It is obvious from the equality of the opposite sides and angles of the parallelogram ($a$)—of the vertical angles at E ($b$)—of the alternate angles at B, D ($c$)—that BD is bisected in E;—and therefore ($d$), that

$AB^2 + AD^2 = 2AE^2 + 2ED^2$, and that

$CB^2 + CD^2 = 2CE^2 + 2EB^2$. Therefore, the squares of the four sides are equal to four squares of the half of each diameter. But four squares of the half equal the square of the whole ($e$): therefore the squares of the two diameters equal the squares of the four sides of the parallelogram.

Q. E. D.

Recite ($a$) p. 34, 1; ($b$) p. 15, 1; ($c$) p. 27, 1; ($d$) p. A, 2; ($e$) p. 8, cor. 2, b. 2.

# BOOK THIRD.

## *Definitions.*

1. Equal circles are those of which the diameters or radii are equal.

2. A tangent is a straight line touching a circle, without cutting it, however produced.

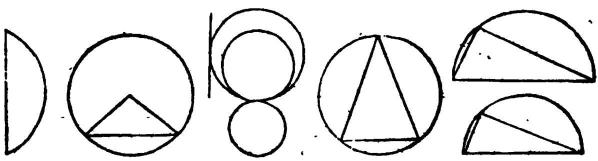
3. A circle touches a circle when they meet and do not cut one another.

4. Chords are equidistant from the centre, when the perpendiculars drawn to them from the centre are equal: when the perpendicular is greater the chord is more remote.

5. The angle of a segment is the declination of its chord from the arc.

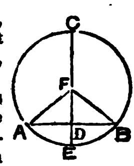
6. An angle is in a segment when the sides containing it are in the segment: and an angle is said to insist, or stand upon the arc intercepted between the sides containing the angle.

7. Similar segments of a circle are those which contain equal angles.



## *Propositions.*

### 1 P. To find the centre of a given circle (ABC).



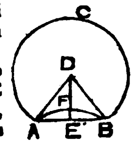

Construction. In the circle draw any chord AB, and bisect it in D (*a*); through D draw CE at right angles to AB (*b*); bisect CE in F (*a*); draw FA, FB (*c*).

Argument. In the triangle ADF, BDF, the sides AD, BD are made equal, and DF is common; also the angles at D are equal, being right angles (*d*): therefore the sides FA, FB are equal (*e*); and, being drawn from the bisectional point of CE, they are radii of the circle (*f*) wherefore the point F is the centre sought.

Recite (*a*) p. 10, 1; (*b*) p. 11, 1; (*c*) post. 1;
(*d*) ax. 10, 1; (*e*) p. 4, 1; (*f*) def. 15, 1.

Corollary. The line which bisects another at right angles in a circle, passes through the centre of a circle.

2 Th. If any two points (A, B) be taken in the circumference of a circle (ABC), the straight line (AB) which joins them, shall fall within the circle.

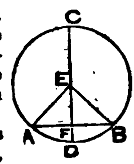



Constr. Find the centre D (*a*), and draw radii from D to A, B, F; also draw DE perpendicular upon AB (*b*).

Argument. In the triangle DAE the exterior angle DEB exceeds the interior DAE (*c*), or its equal DBE (*d*); therefore the radius DB, or its equal DF (*e*), exceeds the side DE (*f*); and so the part DF exceeds the whole DE, which cannot be admitted (*g*). Wherefore AB falls not without the circle; and it cannot fall upon the circumference because it is a straight line (*h*): it therefore falls within the circle. Q. E. D.

Recite (*a*) p. 1, 3; (*b*) p. 12, 1; (*c*) p. 16, 1;
(*d*) p. 5, 1; (*e*) def. 15, 1; (*f*) p. 18, 1;
(*g*) ax. 9, 1; (*h*) def 4, 1.

3 Th. If a diameter (CD) of a circle (ABC) bisect a chord (AB) not passing through the centre, the former will cut the latter at right angles; and the chord so cut shall be bisected (in F).



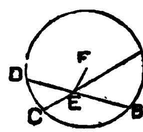

Argument. Find the centre E (*a*), and join EA, EB. Then, in the triangles EAF, EBF the bases AF, BF are equal, by hyp.; EA, EB are equal radii, and EF is common: therefore the adjacent angles AFE, BFE are equal (*b*); and so CD bisects AB at right angles in F (*c*).

Again, since CD cuts AB at right angles, the angles at F are equal (*d*); and so are the angles at A and B, which are opposite to equal radii (*e*); therefore the angles AEF, BEF are equal (*f*), and likewise the sides AF, FB; that is, the chord AB is bisected in F.

Wherefore, if a diameter of a circle, &c. Q. E. D.

Recite (*a*) p. 1, 3; (*b*) p. 8, 1; (*c*) p. 13, 1;
(*d*) ax. 10, 1; (*e*) p. 5, 1; (*f*) p. 26, 32, or 4 b. 1.

4 Th. If in a circle (ABCD) two chords (AC, BD) cut each other (in E), which do not both pass through the centre, they do not bisect each other.




Argument 1. Let F be the centre; then, if AC pass through it, BD cannot pass through F, by hyp.; therefore AC is not bisected by BD (*a*).

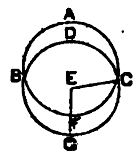
2. If neither of the chords pass through F, draw FE from the centre to the sectional point. Now because FE affects to bisect AC, and also BD, the angles FEA, FEB assume to be right angles (*b*),

and equal to each other (*c*); and so, a part equals the whole, which is impossible (*d*): therefore neither of the chords is bisected in E.

Wherefore, if in a circle, &c. Q. E. D.

Recite (*a*) def. 14, 15, 1; (*b*) p. 3, 3;
(*c*) ax. 10, 1; (*d*) ax. 9, 1.

5 Th. If two circles (ABC, ABG) cut each other, they shall not have the same centre.




Argument. For, if possible, let E be the common centre; and let C be one of their sectional points; join CE, and draw a straight line EFG, to meet the circles in F and G.

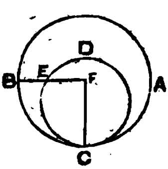
Then EC and EF, also EC and EG, are equal radii (*a*); and so, EF is equal to EG (*b*)—a part equal to the whole, which cannot be admitted (*c*).

Wherefore E is not the common centre; and E is any point whatever.

Hence, if two circles, &c. Q. E. D.

Recite (*a*) def. 15, 1; (*b*) ax. 1; (*c*) ax. 9.

6 Th. If two circles (ABC, CDE) touch each other internally, they shall not have the same centre.




Argument. For, if possible, let F be the common centre, and C the point of contact: join FC, and draw a straight line FEB to meet the circles in E and B.

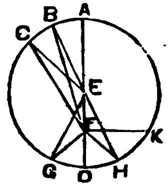
Then FC and FE, also FC and FB are equal radii (*a*); and so, FE and FB are equal (*b*)—a part equal to the whole, which is impossible (*c*).

Wherefore, F is not the common centre; nor is any other point.

Therefore, if two circles, &c. Q. E. D.

Recite (*a*) def. 15, 1; (*b*) ax. 1; (*c*) ax. 9.

7 Th. From any point (F) not the centre, in the diameter (AD) of a circle (ABCD), the greatest straight line drawn to the circumference passes through the centre (E): and FD, the other part of the diameter, is the least: and other lines drawn from that point diminish as they recede from the greatest towards the least: also from the same point, only two equal straight lines can be drawn, one on each side of the diameter.




Argument 1. If straight lines be drawn from E and F in the diameter, to the points B, C, G in the circumference; there will be in each of the triangles BEF, CEF, GEF, a radius, and the side EF common, which are equal to AF: but one side of a triangle is less than the sum of the other two (*a*); therefore FG, FC, or FB is less than FA, which passes through the centre.

2. And, because EG, or its equal ED (*b*), that is EF, FD, is less than EF, FG (*a*), taking away the common part EF, the remainder FD is less than the remainder FG; but FD is the other part of the diameter.

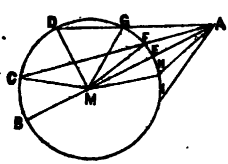
3. Also, since the sides CE, EF, though equal to the sides BE, EF, contain a less angle; therefore (*c*) the base CF is less than the base BF. For like reason the base GF is less than the base CF (*d*).

4. Make now the angle FEH equal to the angle FEG; join FH: then because EG, EH are equal radii, and EF is common; the bases FG, FH are equal (*e*). But, besides FH, no straight line can be drawn from the point F to the circumference equal to FG; for such line, as FK, must incline towards the greatest or least, and be greater or less than FH or FG.

Therefore from any point, not the centre, &c. Q. E. D.

Recite (*a*) p. 20, 1; (*b*) def. 15, 1; (*c*) p. 24, 1;
(*d*) p. 23, 1; (*e*) p. 4, 1.

8 Th. From any point (A) without a circle (BCD), the greatest straight line (AB) that can be drawn to the concave arc passes through the centre (M); and other lines, drawn from the same point, diminish as they decline from the greatest. But of those which meet the convex arc, the least (AE) is the exterior part of the greatest; and the rest increase as they decline from the least: and only two of them can be equal, one on each side of the least.




Argument 1. The distance from A through M to B, C, or D is equal; but the straight line AC is less than the distance from A to C through M (*a*); therefore AB is greater than AC, and it passes through the centre M.

2. Because the angle AMD is less than the angle AMC, the base AD is less than the base AC (*b*), and it is more remote from AB.

3. Again, because AE and EM are less than AF and FM (*a*), and that the radii EM and FM are equal, AE is less than AF, and it is the exterior part of AB.

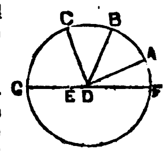
4. Also, since AF, FM are less than AG, GM (*c*); take away the equal radii FM, GM, the remainder AG is greater than AF, and it is more remote from AE.

5. Make now the angle AMH equal to the angle AMF (*d*), and join AH: then in the triangles AFM, AHM, the side AM is common; HM and FM are equal radii, and the angles AMH, AMF are equal; therefore these two triangles are equal, and have the side AH equal to AF (*e*). Moreover, no line but AH can be drawn to the circumference, equal to AF: for such line, as AI, must decline more or less from AE, and be greater or less than AF.

Wherefore, from any point without a circle, &c. Q. E. D.

Recite (*a*) p. 20, 1; (*b*) p. 24, 1; (*c*) p. 21, 1; (*d*), p. 23, 1; (*e*) p. 4, 1.

9 Th. If from a point (D), within a circle (ABC), more than two equal straight lines can be drawn, that point is the centre of the circle.



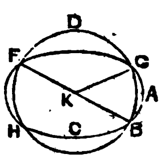

Constr. From the point D draw three equal straight lines DA, DB, DC: then, if D be not the centre, let it be E; and, through E and D, draw FG; which is therefore a diameter of the circle (*a*).

Argument. Because D is a point in the diameter of a circle, not the centre, DG, which passes through the centre, is the greatest straight line that can be drawn from it to the circumference (*b*); and of the rest, DC is greater than DB, and DB than DA: but these three were assumed to be equal; therefore the conditions are inconsistent; and *must* be, while E, or any point except D, is taken as the centre.

Wherefore, if from a point, &c. Q. E. D.

Recite (*a*) def. 16, 1; (*b*) p. 7, 3.

10 Th. One circumference of a circle cannot cut another in more than two points.



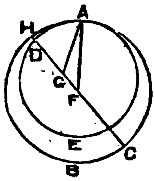

If it be possible, let the circumferences ABC, DEF mutually intersect in *three* points, B, G, F; and let K be the centre of the circle ABC, having KB, KG, KF as equal radii.

Now, from a point K, within the circle DEF, there are drawn to the circumference *more than two* equal straight lines: the point K is therefore the centre of the circle DEF (*a*), as well as that of ABC: and so, two circles which cut one another have the same centre (*b*); which cannot be said of true circles.

Wherefore, one circumference of a circle, &c. Q. E. D.

Recite (*a*) p. 9, 3; (*b*) p. 5, 3.

11 Th. If two circles (ABC, ADE), touch one another internally (in A), the straight line which joins their centres, being produced, shall pass through their point of contact.



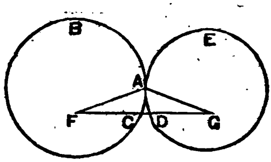

Argument. Let F, G, be their central points: then, if GF which joins them do not pass through A, it must pass otherwise, as through HDC: join AF, AG.

Then FA, FH are equal radii; as also GA, GH (*a*): but FA is less than FG and GA (*b*), or FG and GH, that is, than FH. To be equal and less cannot be admitted: therefore F and G cannot be the true centres, which must be in the same straight line with A, the point of contact.

Wherefore, if two circles, &c. Q. E. D.

Recite (*a*) def. 15, 1; (*b*) p. 20, 1.

12 Th. If two circles (ABC, ADE), touch each other externally (in A), the straight line which joins their centres shall pass through the point of contact.



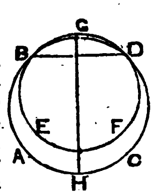

Argument. Let F, G, be their central points: then, if GF which joins them, do not pass through A, it must pass otherwise, as through FCDG: join AF, AG.

Then FA, FC are equal radii; as also GA, GD (*a*): therefore the two FA, GA are equal to the two FC, GD. But FG is greater than FC, GD, and therefore greater than FA, GA; and so, one side of a triangle is greater than the other two; but it is also less (*b*), which is impossible. Therefore F and G are not the true centres, which must be in the same straight line with A, the point of contact.

Wherefore, if two circles, &c. Q. E. D.

Recite (*a*) def. 15, 1; (*b*) p. 20, 1.

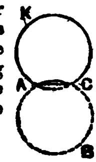
13 Th. One circle cannot touch another in more than one point, whether they touch inside or outside.




Argument 1. If the circles ABC, DEF are said to touch each other internally in two points B, D; join BD; and draw GH to bisect BD at right angles (*q*.) Then, as the points B, D, are in the circumference of each of the circles, the chord BD falls within both (*b*), and their centres are in the diameter GH (*c*); therefore GH passes through their point of contact (*d*), which is neither B, nor D: hence the circles which affect to touch each

other in B, D, are not true; and so, one circle cannot touch another on the inside in more than one point.

2. Again, let the circles ABC, ACK touch each other externally, as in A, C; join AC. Then because the points A, C, are in the circumference of each of the circles, the chord AC falls within both (*b*); and so, while affecting only to touch, the circles intersect each other: therefore one circle cannot touch another on the outside in more than one point.



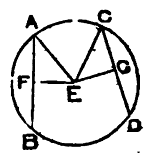

Wherefore, one circle cannot touch, &c.

Q. E. D.

Recite (*a*) p. 10, 11, 1; (*b*) p. 2, 3;
(*c*) cor. p. 1, 3; (*d*) p. 11, 3.

14 Th. Equal chords (AB, CD), in a circle (ABCD), are equidistant from the centre; and chords equidistant from the centre are equal to one another.

Constr. Find the centre E (*a*), from which draw perpendiculars EF, EG, upon AB, CD (*b*): join EA, EC.




Argument 1. The equal chords AB, CD are bisected by the perpendiculars EF, EG, drawn from the centre (*c*). Wherefore, their halves AF, CG are equal (*d*); and the squares of the halves are equal. But the squares of AF, FE, and of CG, GE, are equal to the squares of the equal radii EA, EC, each to each: therefore the squares of AF, FE are equal to the squares of CG, GE (*e*): from these take the equal squares of AF, CG; the remaining squares FE, GE are equal: and so, EF is equal to EG: therefore the chords AB, CD are equidistant from the centre (*f*).

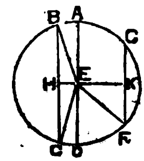
2. And if the central distances EF, EG are equal, the chords AB, CD are also equal: for the squares of the equal radii EA, EC are equal; from which taking the equal squares of EF, EG, the remainders will be equal (*g*), which are the squares of AF, CG: therefore AF is equal to CG; and they are halves of AB, CD, which are therefore equal (*h*).

Wherefore, equal chords in a circle, &c. Q. E. D.

Recite (*a*) p. 1, 3; (*b*) p. 12, 1, and 3, 3; (*c*) cor. p. 1, 3;
(*d*) ax. 7; (*e*) p. 47, 1; (*f*) def. 4, 3;
(*g*) ax. 3; (*h*) ax. 6.

15 Th. The diameter (AD) is the greatest chord in a circle (ABCD); and of all others, the chord (BC) nearer to the centre (E) is greater than one more remote (FG); and the greater is nearer to the centre than the less.

Constr. Draw EH, EK perpendicular on BC, FG (*a*); join EB, EC, EF, (*b*).




Argument 1. The one side BC is less than the two EB, EC (*c*), or their equals EA, ED (*b*), that is, AD, which is therefore greater than BC.

2. The equal radii EB, EF have equal squares, and they subtend right angles at H, K (*d*); therefore the two squares on EH, HB and the two on EK, KF are equal. But EH is less than EK, and its square is the less; therefore the square on HB exceeds that on KF; and so HB is greater than KF. But these are the halves of BC, FG, which are bisected in H, K (*e*); therefore BC is greater than FG.

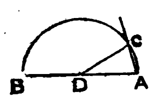
3. And, if BC be greater than FG, it is nearer the centre. For the two squares on EH, HB prove equal to the two on EK, KF, and that on HB exceeds that on KF; therefore, of the two squares which remain, that on EH is less than that on EK; and so, EH is less than EK: therefore BC is nearer the centre than FG (*f*).

Wherefore, the diameter is the greatest chord, &c.

Q. E. D.

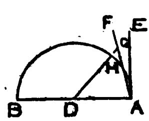
Recite (*a*) p. 12, 1; (*b*) def. 15, 1; (*c*) p. 20, 1;
(*d*) p. 47, 1; (*e*) p. 3, 3; (*f*) def. 4, 3.

16 Th. The straight line (AE) drawn at right angles to the diameter (AB) of a circle, from its extremity, falls without the circle; and no straight line can make an acute angle with that diameter so great, or with the line which falls without the circle so small, as not to cut the circle.




1. For if a straight line, as AC, can be drawn from the point A, at right angles to AB, so as to fall within the circle, draw DC:

Then, because of the equal radii, DAC is an isosceles triangle (*a*); and the angles DAC, DCA are equal (*b*); and both right angles: for DAC is a right angle: add the angle ADC; therefore the three angles of a triangle are greater than two right angles (*c*), which is impossible. Therefore a straight line cannot be drawn from the point A to fall within the circle, and be at right angles with the diameter. Neither can it fall upon the circumference and be a straight line (*d*). It falls therefore without the circle.



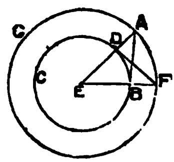

2. Again, if a straight line, as AF, can be drawn between AE and the circumference, make DG perpendicular to AF, cutting the circumference in H: then, because DGA affects to be a right angle, and DAG is acute, the side DA (*e*), or its equal DH, must be greater than DG: but a part is not greater than the whole (*f*). Therefore AF cuts the circle.

Wherefore, the straight line drawn, &c. Q. E. D.

Recite (*a*) def. 24, 1; (*b*) p. 5, 1; (*c*) p. 32, 1;
(*d*) def. 4, 5, 1; (*e*) p. 18, 1; (*f*) ax. 9.

Cor. The straight line drawn at right angles to the diameter or radius, from its extremity, touches the circle, and only in one point; for it would cut the circle if it should meet it in two, by p. 2, 3: and in the same point only one line can touch the circle.

17 P. To draw a straight line from a given point (A), without a circle (BCD), or in the circumference (as at D), which shall touch the circle.



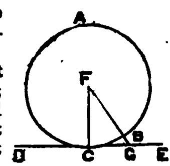

Constr. Find the centre E (*a*); join EA; and upon E, with the radius EA, describe the circle AFG; from the point D draw DF at right angles to EA (*b*); join EBF and AB.

Argument 1. Because of the equal radii (*c*) the triangles AEB, FED have two sides AE, EB in the one, equal to two sides FE, ED in the other, and the angle at E common; therefore (*d*) the remaining sides AB, FD are equal; as also the angles opposite to the equal sides; namely, ABE to FDE: but FDE is a right angle; and so, ABE is the same; and the radius EB meets AB at right angles in the circumference: therefore AB touches the circle (*e*), and it is drawn from the given point A.

2. If the given point be in the circumference, as at D, draw DE to the centre, and DF at right angles to DE: DF is drawn to touch the circle (*e*); which was to be done.

Recite (*a*) p. 1, 3; (*b*) p. 11, 1: (*c*) def. 15, 1;
(*d*) p. 4, 1; (*e*) cor. 16, 3.

18 Th. If a straight line (DE), touch a circle (ABC), the radius (FC) drawn to the point of contact (C) shall be perpendicular to the tangent, or touching line.

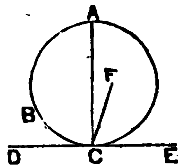



Constr. Find the centre F (*a*); and if FC be not perpendicular to DE, draw FBG perpendicular to it (*b*).

Argument. Because FGC affects to be a right angle, FCG must be less than a right angle (*c*); and FG is therefore less than FC (*d*), or its equal FB (*e*); but the whole is greater than its part (*f*): wherefore FG is not less than FB, nor is it perpendicular to DE; neither is any other line drawn from the centre which does not meet DE in the point of contact. FC is therefore perpendicular to DE. Q. E. D.

Recite (*a*) p. 1, 3; (*b*) p. 12, 1; (*c*) p. 32, 1;
(*d*) p. 18, 1; (*e*) def. 15, 1; (*f*) ax. 9.

19 Th. If a straight line (DE) touch a circle (ABC), and from the point of contact (C) a straight line (AC) be drawn in the circle, at right angles to that tangent, the centre of the circle shall be in that straight line.

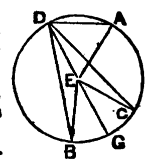



Argument. If the centre of the circle be not in AC, it must be out of it, as in F; join CF. Then because DE touches the circle, and FC affects to be drawn from the centre to the point of contact, FC is perpendicular to DE (*a*), and the angle FCE is a right angle (*b*): but ACE is a right angle; and all right angles are equal to one another (*c*); therefore FCE is equal to ACE: but the whole is greater than its part (*d*); and two magnitudes cannot be at once equal and unequal: therefore F is not the centre; neither is any point on the right or left of AC.

Therefore, if a straight line, &c. Q. E. D.

Recite (*a*) p. 18, 3; (*b*) def. 10, 1;
(*c*) ax. 10; (*d*) ax. 9.

**20 Th.** The angle at the centre of a circle is double the angle at the circumference upon the same arc.




The angles AEB, BEC, AEC are central:

The angles ADB, BDC, ADC are at the circumference, and upon the same arcs as the former. Draw the diameter DEG.

Because of the equal radii, the triangles EAD, EBD, ECD, are isosceles (*a*), and have equal angles opposite to the equal sides (*b*).

And, because the side DE is produced, the exterior angle AEG is equal to the two interior opposite angles EAD, EDA (*c*), or double EDA; the exterior BEG to the two interior EBD, EDB, or double EDB; and the exterior CEG to the two interior ECD, EDC, or double EDC.

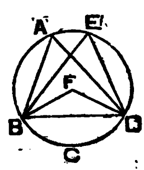
Therefore, the sum of AEG and BEG is double the sum of ADG and BDG; that is, AEB is double ADB.

And the difference of AEG and CEG is double the difference of ADG and CDG; that is, AEC is double ADC.

Wherefore, the angle at the centre, &c. Q. E. D.

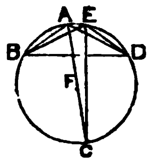
Recite (*a*) def. 24, 1; (*b*) p. 5, 1; (*c*) p. 32, 1.

**21 Th.** The angles in the same segment of a circle are equal to one another.




Let BAED be a segment: the angles BAD, BED are equal to one another.

1. If the centre F be in the segment, join FB, FD: then because the angle BFD is at the centre and the angles BAD, BED at the circumference. on the same arc, each of the latter is half the former (*a*);—and halves of the same are equals (*b*): therefore BAD is equal to BED.

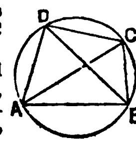



2. If the centre F be not in the segment, draw the diameter AFC, and join CE: then, the angles BAC, BEC, in the same segment are equal, by case 1; and the angles CAD, CED are equal, for the same reason: therefore, BAC and CAD are equal to BEC and CED; that is, BAD is equal to BED.

Wherefore, the angles in the same segment, &c.
Q. E. D.

Recite *(a)* p. 20, 3; *(b)* ax. 7.

22 Th. The opposite angles of any quadrilateral figure inscribed in a circle (ABCD) are together equal to two right angles.




In the quadrilateral figure ABCD, the two angles ABC, ADC are equal to two right angles: join AC and BD.

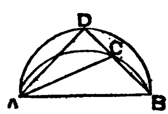
In the triangle ABC, the three angles are equal to two right angles *(a)*: but the angles ACB, ADB, as also the angles BAC, BDC are equal *(b)*; therefore, the two ACB and BAC are equal to the two BDC and ADB; that is, to ADC: to each side add ABC; then the three ACB, BAC, ABC are equal to the two ADC and ABC; but the three are equal to two right angles; and so, the two are equal to the same.

In this manner, it may be shown that BAD and BCD are also equal to two right angles.

Therefore, the opposite angles, &c. Q. E. D.

Recite *(a)* p. 32, 1; *(b)* p. 21, 3.

23 Th. Upon one side of the same chord (AB) no two similar segments of circles can be described, which shall not coincide with each other.




If possible, let the segments ACB, ADB, which are on the same side of AB, and do not coincide, be similar.

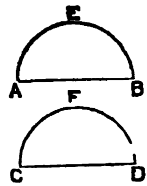
Then, since the circles, of which these are segments, can cut each other only in two points *(a)*; the segments may meet each other only in the points A, B. Draw the straight line BCD; and join AC, AD.

Now similar segments contain equal angles *(b)*; therefore the angles ACB, ADB are equal: but ADC is a triangle, and its exterior angle is ACB, which exceeds the interior ADB, *(c)*: therefore, since one angle cannot be equal to, and greater than another, the segments ACB, ADB are not similar.

Therefore, upon one side of the same chord, &c. Q. E. D.

Recite *(a)* p. 10, 3; *(b)* def. 7, 3; *(c)*; p. 16, 1.

**24 Th.** Similar segments of circles (AEB, CFD), upon equal chords (AB, CD), are equal to each other.

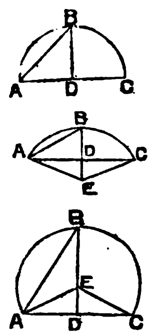



Argument. If the segment AEB be applied to the segment CFD, because the chords are equal (*a*), their extreme points A, B, and C, D, shall coincide: but the same points are the extremities of the arcs, which must also coincide; because the segments are similar (*b*): therefore, the perimeters everywhere coincide and bound the same space (*c*).

Therefore, similar segments, upon equal chords, &c. Q. E. D.

Recite (*a*) def. 3, 1; (*b*) def. 7, and p. 23, 3; (*c*) def. 37, 1.

**25 P.** A segment of a circle (ABC) being given, to find the centre, or to describe the circle.

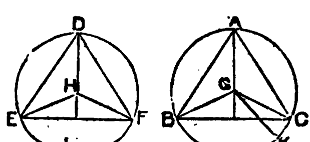



1. Bisect AC in D (*a*); through D, draw DB at right angles to AC (*b*); join AB: then if the angles BAD, ABD be equal, the sides DA, DB are equal (*c*), and D is the centre sought.

2. If DB be less, or greater than DA, or DC, make the angle BAE equal to the angle ABE (*d*); then EA equals EB (*c*); join EC. And, because of the right angles at D; and that DA, DE are equal to DC, DE, the bases EA, EC are equal (*e*). Therefore, since three equal straight lines, EA, EB, EC, are drawn from the point E to the circumference, E is the centre of the circle (*f*), of which ABC is a segment.

Recite (*a*) p. 10, 1; (*b*) p. 11, 1; (*c*) p. 6, 1; (*d*) p. 23, 1; (*e*) p. 4, 1; (*f*) p. 9, 3.

**26 Th.** In equal circles (ABC, DEF) equal angles at the centres (BGC, EHF), or at the circumferences (BAC, EDF), stand upon equal arcs.




Argument. Join BC, EF; then, since the circles are equal, the radii are also equal (*a*); and since the angles BGC, EHF contained by the equal radii, are equal, the base BC is equal to the base EF (*b*).

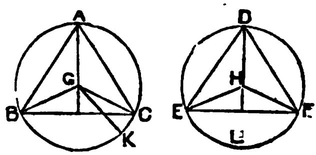
Again, because the angles at A and D are equal, the segments containing them are similar (*c*); also, because the chords are equal, the segments containing the equal angles are equal (*d*): therefore, if the equal segments be taken from

the equal circles, the segments left will be equal (*e*): and so, the segments BKC, ELF are equal: and the arc BKC is equal to the arc ELF.

Wherefore, in equal circles, equal angles, &c. Q. E. D.

Recite (*a*) def. 15, 1; (*b*) p. 5, 1; (*c*) def. 7, 3;
(*d*) p. 24, 3; (*e*) ax. 3.

27 Th. In equal circles (ABC, DEF), equal arcs (BKC, ELF) subtend equal angles, whether at the centres (BGC, EHF), or at the circumferences (BAC, EDF).

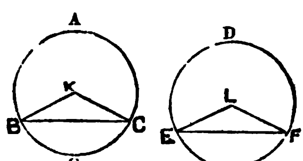



Argument. If the central angles be equal, their halves at the circumferences will be equal (*a*); but if unequal, one of them, as BGC, is the greater: make BGK equal to EHF (*b*). Now equal angles stand upon equal arcs (*c*); therefore, the arc BK is equal to the arc ELF, or its equal BKC: so the part is equal to the whole; but it is not (*d*). Therefore, BK is not, but BKC is equal to ELF; and the angles BGC, EHF, at the centres, are equal; and their halves, A and D, at the circumferences, are also equal.

Wherefore, in equal circles, equal arcs, &c. Q. E. D.

Recite (*a*) p. 20, 3; (*b*) p. 23, 1;
(*c*) p. 26, 3; (*d*) ax. 9, 1.

28 Th. In equal circles (ABC, DEF) equal chords (BC, EF) cut off equal arcs; the greater equal to the greater, and the less to the less.




Let the greater segments be BAC, EDF, and the less BGC, EHF. Find the centres K, L (*a*); and draw radii to B, C, and to E, F.

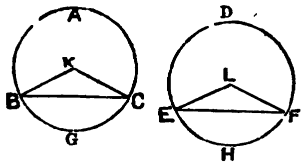
Then since the circles are equal, the radii are equal (*b*); and, since the chords are equal, the angles at K, L, are also equal (*c*). But equal angles stand upon equal arcs (*d*); therefore, the arc BGC is equal to the arc EHF.

Again, taking the arcs now proved equal, from the equal circumferences, the remaining arcs are equal; namely, BAC to EDF (*e*).

Wherefore, in equal circles, equal chords, &c. Q. E. D.

Recite (*a*) p. 1, 3; (*b*) def. 15, 1; (*c*) p. 8, 1;
(*d*) p. 26, 3; (*e*) ax. 3, 1.

**29 Th.** In equal circles (ABC, DEF), equal arcs are subtended by equal chords.



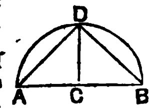

Let the arcs BGC, EHF be equal; the chords BC, EF shall also be equal; join K to B and C, and L to E and F.

Then, because the arcs are equal, the central angles are equal (*a*); and because the circles are equal, the radii are equal (*b*); and, since the triangles BKC, ELF have two sides, and the contained angle in the one equal to the same in the other, their bases are equal (*c*); namely, the chords BC, EF.

Wherefore, in equal circles, equal arcs, &c. Q. E. D.

Recite (*a*) p. 27, 3; (*b*) def. 15, 1; (*c*) p. 4, 1.

**30 P.** To bisect a given arc (ADB); that is, to divide it into two equal parts.

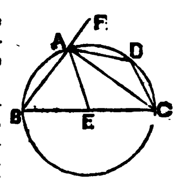



Constr. Draw the chord AB, and bisect it in C (*a*); draw CD at right angles to AB (*b*); join DA, DB.

Argument. Because of the right angles at C, and that CA, CD are equal to CB, CD, the triangles CAD, CBD have their bases equal (*c*); namely, the chords AD and BD. Now these equal chords cut off equal arcs (*d*), less than semicircles; because DC produced is a diameter (*e*): therefore the given arc is bisected in D, as required.

Recite (*a*) p. 10, 1; (*b*) p. 11, 1; (*c*) ax. 10 and p. 4, 1; (*d*) p. 28, 3; (*e*) p. 15, 3.

**31 Th.** In a circle, the angle in a semicircle is a right angle, in a greater segment the angle is acute; in a less segment the angle is obtuse.




Let ABCD be a circle, BC a diameter, E the centre, BAC a semicircle, ABC the greater segment, and ADC the less. Join AE, and produce BA to F.

1. The radii EA, EB, EC are opposite to equal angles (*a*); that is, the angles EAB, EBA, EAC, ECA are equal to each other; and they are equal two and two: therefore EBA and ECA are equal to EAB and EAC, or the whole angle BAC. But if one angle of a triangle be equal to the other two, it must be a right angle (*b*); therefore BAC, an angle in a semicircle, is a right angle, as stated.

2. Again, since in the triangle ABC, the two angles EBA, ECA are equal to the right angle BAC, each of them is less than a right an-

gle: therefore ABC, the angle in the greater segment, is less than a right angle.

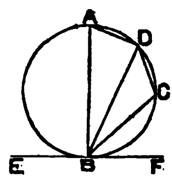
3. Also, the quadrilateral ABCD, being inscribed in a circle, any two of its opposite angles are equal to two right angles (*c*); therefore, since ABC proves to be acute, its opposite ADC is obtuse, and it is in the less segment.

Wherefore, in a circle, the angle, &c. Q. E. D.

Recite (*a*) p. 5, 1; (*b*) p. 32, 1; (*c*) p. 22, 3.

Cor. If an angle of a triangle be equal to its adjacent angle, it is equal to the other two (p. 32, 1), and is therefore a right angle.

## 32 Th. If a straight line (EF) touch a circle (ABC), and from (B), the point of contact, any chord be drawn in the circle, the angles thus made, on either side, shall be equal to the angles in the alternate segment.




1. If the chord BA pass through the centre (*a*), each of the segments will be a semicircle, and shall contain a right angle, equal to ABE, or ABF (*b*).

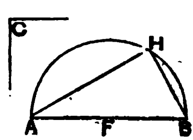
2. If the chord BD make oblique angles with EF (*c*); join AD: then BDA is an angle in a semicircle, and therefore a right angle, equal to ABF, as before; or to the sum of BAD and ABD (*d*): from each of these equals take ABD, the remainders DBF and DAB are equal (*e*); and the latter is in the alternate segment made by DB.

3. At any point C, in the less segment, make an angle BCD; then, since ABCD is a quadrilateral inscribed in a circle, its opposite angles are equal to two right angles (*f*); therefore BAD, BCD are together equal to DBF, DBE (*g*): but DBF proves equal to DAB; therefore DBE is equal to BCD, which is in the alternate segment.

Wherefore, if a straight line touch, &c. Q. E. D.

Recite (*a*) p. 19, 3; (*b*) p. 31, 3; (*c*) Note def. 8, 1;
(*d*) cor. 31, 3; (*e*) ax. 3, 1; (*f*) p. 22, 3;
(*g*) p. 13, 1.

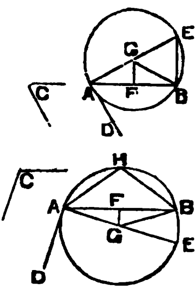
## 33 P. Upon a given straight line (AB), to describe a segment of a circle, containing an angle equal to a given rectilineal angle (C).




In every case, whether the given angle C be right, or oblique, bisect the given line AB in the point F (*a*).

1. Let C be a right angle: then, upon F as centre, describe the semicircle AHB; join H to A and B. The angle at H, being in a semicircle, is a right angle (*b*), and therefore equal to C (*c*).

2. Let C be any oblique angle (*d*): make, at the point A, angles BAD, equal to C, and DAE a right angle (*e*); draw FG at right angles to AB (*f*); join GB.



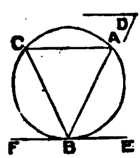

Now, on account of the right angles at F, and that FA, FG are equal to FB, FG, the bases GA, GB are equal (*g*); and about G as centre, a circle may be drawn to pass through the points A, E, B, of the greater segment, and A, H, B, of the less; and DA shall touch the circle in A (*h*). Therefore the angle AEB is equal to BAD, or C, acute; and the angle AHB is equal to BAD, or C, obtuse.

Therefore, upon the given straight line, a segment has been described, containing an angle given in variety of magnitude, as right, acute, and obtuse.

Q. E. F.

Recite (*a*) p. 10, 1; (*b*) p. 31, 3; (*c*) ax. 10, 1; (*d*) Note def. 8, 1; (*e*) p. 23, 1; (*f*) p. 11, 1: (*g*) p. 4, 1, and cor. p. 1, 3; (*h*) p. 31, 32, and cor. p. 16, 3.

34 P. To cut off a segment from a given circle (ABC), which shall contain an angle equal to a given rectilineal angle (D).



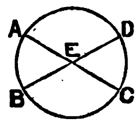

Constr. Draw any tangent, as EF, touching the given circle in a point B (*a*); and at the point B, in the straight line FB, make the angle FBC equal to the angle D (*b*).

Argument. Because EF touches the circle; and from B, the point of contact, a chord BC is drawn in the circle; there are two angles equal to the two in the alternate segments (*c*): therefore the angles FBC and BAC are equal; but FBC was made equal to the given angle D; and so, BAC is equal to D (*d*); and it is in a segment cut off from a given circle; which was to be done.

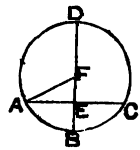
Recite (*a*) p, 17, 3; (*b*) p. 23, 1; (*c*) p. 32, 3; (*d*) ax. 1.

35 Th. If two chords (AC, BD) in a circle cut each other, the rectangle of the segments (AE, EC) of one of the chords, is equal to the rectangle of the segments (BC, CD) of the other.

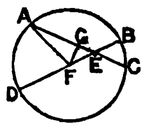
1. If the sectional point E, be the centre of the circle; then it is obvious that AE×EC equals BE×ED: for the segments are equal radii (*a*).



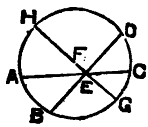

2. If the chords cut each other at right angles in E, but not in the centre, one of them AC is bisected in E (*b*), the other passes through the centre F, where it is bisected (*c*): join AF. Now, since BD is divided equally in F, and unequally in E, $BF^2$, or $AF^2 = BE \times ED + EF^2$ (*d*) $= AE^2 + EF^2$ (*e*): from each of these equals take $EF^2$; then $BE \times ED = AE^2$, $AE \times EC$: for AE equals EC.




3. If BD pass through the centre F, and cut AC obliquely in E, draw FG perpendicular to AC (*b*); join AF: therefore, since BD is bisected in F, and AC in G, and both unequally divided in E, $BF^2 = BE \times ED + EF^2$ (*d*); and $EF^2 = EG^2 + GF^2$ (*e*); $AG^2 = AE \times EC + EG^2$ (*d*); to these add $GF^2$ (*f*); then, $AG^2 + GF^2 = AE \times EC + EG^2 + GF^2 = AF^2$ (*e*) $= BF^2$: therefore, $BE \times ED + EG^2 + GF^2 = AE \times EC + EG^2 + GF^2$; from which equals taking $EG^2 + GF^2$, which are common, $BE \times ED = AE \times EC$ (*f*).




4. If neither AC nor BD pass through the centre F, draw the diameter GEFH; then since either of the rectangles AE×EC, or BE×ED proves equal to GE×EH, as above; in this case also AE×EC = BE×ED (*f*).



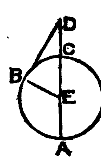

Wherefore, if two chords in a circle, &c.

Q. E. D.

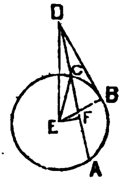
Recite (*a*) def. 15, 1; (*b*) p. 3, and cor. p. 1, 3; (*c*) def. 14, 1; (*d*) p. 5, 2; (*e*) p. 47, 1; (*f*) ax. 1, 2, 3.

36 Th. If from any point (D) without a circle (ABC) two straight lines be drawn, one of which (DCA) cuts, and the other (DB) touches the circle, the rectangle of the secant and its exterior part is equal to the square of the tangent.

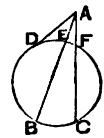
1. If the secant DCA, pass through the centre of the circle E, join EB: then EBD is a right angle (*a*); and $DE^2 = DB^2 + BE^2$ (*b*); and because AC is bisected in E and produced to D, $DE^2 = AD \times DC + CE^2$ (*c*); therefore $DB^2 + BE^2 = AD \times DC + CE^2$ (*d*): take now from both sides the squares of the radii BE, CE, the remainders are equal (*e*); namely, $DB^2 = AD \times DC$.



2. If the secant DCA pass on one side of the centre E; join EB, EC, ED, and draw EF perpendicular to AD ($f$); therefore $DE^2=DF^2+FE^2$ ($b$)$=DB^2+BE^2$, as above: and because AC is bisected in F and produced to D, $DF^2=AD+DC \quad CF^2$ ($c$); to both sides add $FE^2$; therefore $DF^2+FE^2=AD\times DC+CF^2+FE^2$ ($g$): but $CF^2+FE^2=CE^2$ ($b$), or $BE^2$; substituting the latter, $DF^2+FE^2=AD\times DC+BE^2$; therefore $DB^2+BE^2=AD\times DC+BE^2$: take from these $BE^2$, which is common, $DB^2=AD\times DC$ ($e$).




Cor. If from any point A, without a circle, two secants AEB, AFC be drawn; the rectangle of the one AB, and its exterior part AE, is equal to the rectangle of the other AC, and its exterior part AF: for each of these rectangles is equal to the square of the tangent AD ($d$).



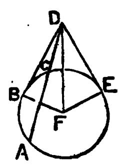

Wherefore, if from any point without, &c.

Q. E. D.

Recite ($a$) p. 18, 3; ($b$) p. 47, 1; ($c$) p. 6, 2; ($d$) ax. 1; ($e$) ax. 3; ($f$) p. 12, 1; also p. 3, 3; ($g$) ax. 2.

37 Th. If from a point (D), without a circle (ABC), two straight lines be drawn, one (DB) meeting, and the other (DEA) cutting the circle; if the rectangle of the cutting line and its exterior part (DC), be equal to the square of the line which meets the circle, the latter line shall touch and not cut the circle.

Draw the tangent DE ($a$); find the centre of the circle F ($b$); join FE, FD, FB: then FED is a right angle ($c$): and because DE touches and DCA cuts the circle, $DE^2=AD\times DC$ ($d$); but $DB^2$ is given equal to the same rectangle; therefore $DB^2$ equals $DE^2$, and DE is equal to DB; the radii FB, FE are also equal, and FD is common to the two triangles DBF, DEF; which have therefore three sides in the one equal to three sides in the other; therefore ($e$) the angle DBF is equal to the angle DEF; and the latter being a right angle by construction, the former is also a right angle; and so, the straight line DB touches the circle ($f$).

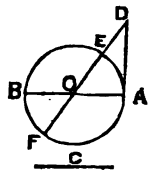



Therefore, if from a point without a circle, &c.

Q. E. D.

Recite ($a$) def. 2, 3; ($b$) p. 1, 3; ($c$) p. 18, 3; ($d$) p. 36 3; ($e$) p. 8, 1; ($f$) p. 16, 3 and cor.

38 P. **To construct a rectangle equal to a given square, having the difference of its adjacent sides equal to a given line.**




Let C be the side of a given square, and AB the difference of the sides of an equivalent rectangle.

Describe a circle on the diameter AB; at the extremity A, of the diameter, draw the tangent AD, equal to C; through the centre O, draw DF, cutting the circle in E, F. Then the rectangle DE$\times$DF equals the square of AD ($a$), or its equal C; and the difference of DE and DF is EF, which is equal to AB.

Recite ($a$) p. 36 of b. 3

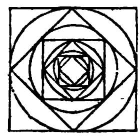
# BOOK FOURTH.

## *Definitions.*

1. One rectilineal figure is said to be inscribed in another, when all the angles of the former are upon the sides of the latter.

Note.—Regular polygons, which have the same number of equal sides, thus inscribed, form a series, and have a certain ratio to each other.

2. One rectilineal figure is described about another, when all the sides of the former pass through the angular points of the latter.



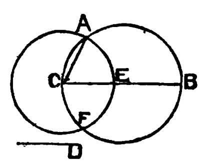
3. A rectilineal figure is inscribed in a circle, when the angles of the former are all in the circumference of the latter. And in this case, the circle is said to be described about the rectilineal figure.

4. A rectilineal figure is described about a circle, when each side of the former touches the circumference of the latter. And in this case, the circle is said to be inscribed in the rectilineal figure.

5. A straight line is said to be placed in a circle, when its extreme points are in the circumference.

## *Propositions.*

### 1 P. In a given circle (ABC) to place a chord equal to a given straight line (D), not greater than the diameter.

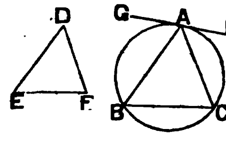



Draw BC, the diameter of the circle: then if the given straight line D, be equal to BC, there is placed in the circle such a chord as was required.

But if D be not equal to BC, it cannot be greater (*a*); make CE equal to D (*b*); and with CE as radius, upon the centre C, describe the circle AEF (*c*); join CA (*d*): therefore, in the circle AEF, CA and CE are equal radii (*e*): but CE is equal D; therefore CA is equal to D; and it is a chord (*f*), not greater than the diameter, placed in the circle ABC; which was to be done.

Recite (*a*) p. 15, 3; (*b*) p. 3, 1; (*c*) pos. 3;
(*d*) pos. 1; (*e*) def. 15, 1; (*f*) def 18. 1.

2 P. In a given circle (ABC), to inscribe a triangle equiangular to a given triangle (DEF).

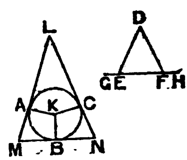



Construction. Draw the tangent GAH (*a*); at A, the point of contact, make the angle HAC equal to the angle E, also the angle GAB equal to the angle F (*b*); join BC.

Argument. Because the straight line GAH touches the circle ABC, and from the point of contact A, the straight lines AB, AC are drawn, the angle HAC is equal to B, and the angle GAB is equal to C, in the alternate segments of the circle (*c*). But HAC is made equal to E, and GAB to F, of the given triangle; therefore, the angle B is equal to E, and the angle C is equal to F, (*d*); so the third angles BAC and D must be equal (*e*). Therefore, the triangle ABC has all its angles equal to those of the given triangle DEF, each to each; and it is inscribed in the given circle ABC, which was to be done.

Recite (*a*) p. 17, 3; (*b*) 23, 1; (*c*) p. 32, 3;
(*d*) ax. 1; (*e*) p. 32, 1.

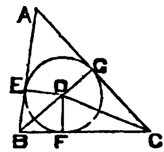
3 P. About a given circle (ABC), to describe a triangle equiangular to a given triangle (DEF).




Produce EF both ways to the points G, H: find K, the centre of the given circle (*a*); draw any radius KB; at the point K, in BK, make angles BKA, BKC equal to the exterior angles DEG, DFH, each to each (*b*); draw tangents through the points A, B, C (*c*), to meet in the points M, L, N. Therefore LM, LN, MN, meet the radii at right angles in the points A, B, C (*d*): and because the quadrilateral AMBK may be divided into two right angled triangles, whose angles are equal to four right angles (*e*); and that two of its angles at A and B, are right angles, the other two, AKB, AMB, are equal to two right angles; but the two angles DEF, DEG are also equal to two right angles (*f*), and BKA was made equal to DEG; therefore, the remaining angles DEF, AMB are equal. It may be proved, in this way, that the angles DFE and N are also equal; and so the third angles D and L must be equal (*e*). Wherefore, about a given circle a triangle is described equiangular to a given triangle; which was to be done.

Recite (*a*) p. 1, 3; (*b*) p. 23, 1, (*c*) p. 17, 3;
(*d*) p. 18, 3; (*e*) p. 32, 1; (*f*) p. 13, 1.

## 4 P. To inscribe a circle in a given triangle (ABC).

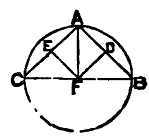



Constr. Bisect the angles B and C (*a*), by straight lines BD, CD, meeting in D, from which point draw perpendiculars to meet the sides of the triangle in the points E, F, G (*b*).

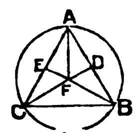
Argument. The triangles BDE, BDF are equal, for the following reasons, viz. The angles EBD, FBD, are halves of the angle EBF (*c*); BED, BFD are right angles (*d*), and BD is common to the two triangles: therefore DE is equal to DF (*e*): for like reason, DG is also equal to DF, or DE; and the circle described upon the centre D, at the distance of any of them, will pass through the points E, F, G; and be inscribed in the given triangle (*f*); which was to be done.

Recite (*a*) p. 9, 1; (*b*) p. 12, 1; (*c*) ax. 7;
(*d*) ax. 10; (*e*) p. 26, 1; (*f*) def. 4, 4.

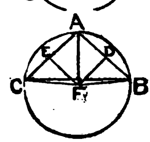
## 5 P. To describe a circle about a given triangle (ABC).




Constr. In points D, E, bisect the sides AB, AC, of the given triangle (*a*); draw DF, EF, at right angles to the sides (*b*); and, if the point F be on the side BC, join FA; if within or without the triangle, join also FB, FC.




Argument. Now, on account of the right angles at D (*c*), and that DA, DF are equal to DB, DF, the bases AF and BF are equal (*d*). In like manner it may be proved that FC is equal to FA, or FB: therefore F is the centre of a circle passing through the three points of the triangle, as required (*e*).



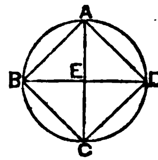

Recite (*a*) p. 10, 1; (*b*) p. 11, 1;
(*c*) ax. 10, 1; (*d*) p. 4, 1;
(*e*) p. 9, 3.

Corollary. The point F is within, on the side, or without the triangle, according as it may be acute, right, or obtuse angled.

## 6 P. To inscribe a square in a given circle (ABCD).

Draw, in the circle, the diameters AC, BD, at right angles to each other (*a*); they will divide the circumference into four equal arcs (*b*); draw also the chords AB, AD, CB, CD, which are all

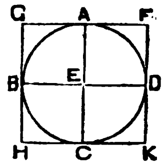
equal to each other (*c*). Now each of the angles made by the chords, at A, B, C, D, is in a semicircle, and is therefore a right angle (*d*). Wherefore, the quadrilateral ABCD, has four equal sides and four right angles, and is therefore a square (*e*), inscribed in the given circle.




Recite (*a*) def. 10, 1; (*b*) p. 26, 3;
(*c*) p. 29, 3; (*d*) p. 31, 3;
(*e*) def. 30, 1.

7 P. To describe a square about a given circle (ABCD).

Draw the diameters AC, BD, at right angles to each other (*a*); also tangents (*b*) to the circle, through the points A, B, C, D, at right angles to the diameters, meeting in the points F, G, H, K.

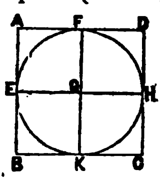



Now, because of the equal alternate angles (*c*) GBD, BDK, the side GH is parallel to FK: for the same reason GF is parallel to HK: therefore FGHK is a parallelogram, whose opposite sides and angles are equal (*d*): but because any one of the angles at E and an interior angle on the same side of those at the points A, B, C, D, are equal to two right angles (*e*); therefore each of the angles F, G, H, K, is a right angle. Wherefore, since the equal sides touch the circle and meet at right angles, there is a square described about the circle as was required (*f*).

Recite (*a*) cor. p. 1, 3; (*b*) def. 2, 3;
(*c*) p. 27, 1; (*d*) p. 34, 1;
(*e*) p. 29, 1; (*f*) def. 30, 1, and 2, 4.

8 P. To inscribe a circle in a given square (ABCD).

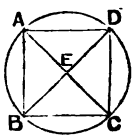
Bisect two sides AB, AD, of the given square, in the points E, F (*a*); draw EH, FK parallel to AB, AD (*b*): then the interior angles A, E and also A, F, are equal to two right angles (*c*): for the same reason all the angles about the point G are right angles; therefore the opposite sides GE, GF, GH, GK are equal (*d*), and G is the centre of a circle that shall touch the sides of the given square, in E, F, H, K (*e*), and be inscribed therein, as required (*f*).




Recite (*a*) p. 10, 1; (*b*) p. 31, 1; (*c*) p. 29, 1;
(*d*) p. 34, 1; (*e*) def. 3, 3; (*f*) def. 14, 1.

9 P. To describe a circle about a given square (ABCD).

Join the opposite angles of the given square by the diameters AC,



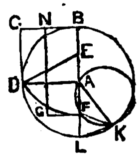

BD, intersecting each other in E. Then the triangles ABC, ADC, have two sides AB, AD, equal, and AC common; also the bases BC, DC are equal; therefore the angle CAB equals the angles CAD (*a*).

In like manner, it is shown, that the other angles of the square are bisected by the straight lines AC, BD, (*b*): but the angles of the square are all equal to each other (*c*); therefore the halves BDA, BDC, ACB, ACD, DBA, DBC are also equal to each other (*d*); and their bases EA, EB, EC, ED are equal (*e*), and E is the centre of a circle that shall pass through the angular points A, B, C, D, (*f*).

Wherefore, a circle has been described about a given square; which was to be done.

Recite (*a*) p. 8, 1; (*b*) p. 9, 1; (*c*) def. 30, ax. 10, 1; (*d*) ax. 7, 1; (*e*) p. 4, 1; (*f*) p. 9, 3.

10 P. To describe an isosceles triangle (ALK), having each of the angles (L, K) at the base double of the angle (A) contained by the equal sides (AL, AK).



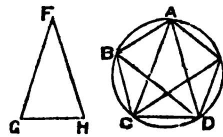

Constr. With any radius AB, and centre A, describe the circle BDLK (*a*); produce BA to meet the circumference in L (*b*); upon AB describe the square ABCD (*c*); bisect AB in E (*d*), and join ED; make EF equal to ED (*e*); place AF in the circle from L to K (*f*); join AK, FK (*g*); about the triangle AFK describe a circle (*h*).

Argument. The radius AL is so divided in F, that the rectangle AL×LF equals the square of AF (*i*); but AF is equal to LK (*k*), which meets the circle AFK in the point K, and AL cuts the same circle in F (*l*); therefore the square of LK equals the rectangle AL×LF, and LK touches the circle AFK (*m*). Now ALK is an isosceles triangle, because AL, AK are equal radii (*n*); therefore the angles ALK, AKL are equal (*o*). Again, because LK touches the circle AFK, and from the point of contact KF is drawn in the circle, the angle FKL is equal to FAK in the alternate segment (*p*); but the angle AKL equals FKL and FKA, or FAK and FKA together: and the exterior angle KFL equals the same FAK and FKA (*q*); therefore KFL equals AKL, or ALK (*r*), and KFL is an isosceles triangle, in which KL and KF are equal (*n*): but KL is made equal to AF (*b*); therefore KF equals AF (*r*), and AKF is an isosceles triangle (*n*), having the angles FKA, FAK equal (*o*); but AKL, or ALK proves equal to both, or double the angle FAK, or LAK.

Wherefore, an isosceles triangle has been described, &c. Q. E. F.

Recite (*a*) pos. 3; (*b*) pos. 2; (*c*) p. 46, 1; (*d*) p. 10, 1; (*e*) p. 3, 1; (*f*) p. 1, 4; (*g*) pos. 1; (*h*) p. 5, 4; (*i*), p. 11, 2; (*k*) p. constr.; (*l*) p. 36, 3; (*m*) p. 37, 3; (*n*) def. 24, 1; (*o*) p. 5, 1; (*p*) p. 32, 3; (*q*) p. 32, 1; (*r*) ax. 1.

11 P. To inscribe an equilateral and equiangular pentagon in a given circle (ABCDE).



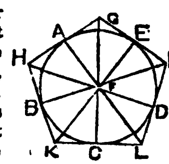

Describe an isosceles triangle FGH (*a*), whose equal sides contain an angle F, half as great as the angle G, or H, at the base. Then in the circle inscribe a triangle ACD, equiangular to FGH; so that the angle C, or D, shall be double the angle A (*b*). Bisect the angles ACD, ADC by the chords CE, DB (*c*); and join AB, BC, CD, DE, EA. ABCDE is the pentagon required.

Because each of the angles ACD, ADC is double of the angle CAD; and are bisected by the chords EC, BD, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another: but equal angles stand upon equal arcs (*d*); therefore the five arcs AB, BC, CD, DE, EA are equal to each other. Again, equal arcs subtend equal chords (*e*); therefore the five chords AB, BC, CD, DE, EA are equal to each other: wherefore the pentagon is equilateral.

It is also equiangular; because the arc AB equals the arc DE; add to each BCD; the sum ABCD is equal to the sum BCDE: and the angle AED stands on the arc ABCD; and the angle BAE on the arc BCDE; therefore the angles AED, BAE are equal (*f*); for the same reason, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED; wherefore the pentagon is equiangular. And thus an equilateral and equiangular pentagon has been inscribed in a given circle: which was to be done.

Recite (*a*) p. 10, 4; (*b*) p. 2, 4; (*c*) p. 9, 1;
(*d*) p. 26, 3; (*e*) p. 29, 3; (*f*) p. 27, 3

12 P. To describe an equilateral and equiangular pentagon about a given circle (ABCDE).




Constr. Let A, B, C, D, E, be the angular points of an inscribed pentagon, as in the last (*a*); so that the arcs AB, BC, CD, DE, EA are equal: then through the points A, B, C, D, E, draw the tangents GH, HK, KL, LM, MG (*b*). If these tangents be equal, and their angles equal, the required pentagon is described about the circle. Find the centre F, and draw the radii FB, FC, FD; join FK, FL.

Argument. In the triangles FBK, FCK, the angles at B, C are right angles (*c*); and the common side FK subtends them: therefore the squares of FB, BK, and also of FC, CK are equal to the square of FK (*d*), and therefore equal to each other (*e*). But the squares of the equal radii FB, FC, are equal; therefore the remaining squares of BK, CK are equal; and BK is equal to CK (*f*). For the same reason CL and DL are equal: therefore the triangles FBK, FCK, FDL, FCL have two sides in each equal to two sides in every other,

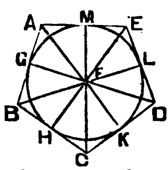
and the angles contained by the two sides equal; therefore the other angles are equal, each to each, to which the equal sides are opposite (*g*); and so, FKB, FKC, FLC, FLD are equal; and any two of them are equal to any other two: therefore FKB, FKC are equal to FLC, FLD; that is, HKL is equal to KLM.

In like manner, it may be shown that each of the angles at H, G, M, is equal to HKL, or KLM: therefore the pentagon is equiangular.

Again, because the perpendicular FC bisects KL, the perpendiculars FB, FD bisect HK and LM: for they intercept equal arcs: and since the halves BK, KC, LD are equal, the wholes HK, KL, LM are equal. In like manner, it may be shown, that GH, or GM is equal to HK, KL, or LM. The pentagon is therefore equilateral; and it proves also to be equiangular, and is inscribed in the given circle ABCDE; which was to be done.

Recite (*a*) p. 11, 4; (*b*) def. 2, p. 16 and cor., also p. 17, of b. 3; (*c*) def. 10, 1; (*d*) p. 47, 1; (*e*) ax. 1; (*f*) ax. 3; (*g*) p. 4, 1.

## 13 P. To inscribe a circle in a given equilateral and equiangular pentagon (ABCDE).



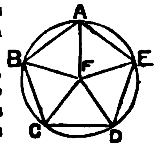

Constr. Bisect the angles BCD, CDE, by the straight lines CF, DF; and from the point F, in which they meet, draw FB, FA, FE.

Therefore, since, in the triangles BCF, DCF, BC, CF are equal to DC, CF, and they contain equal angles, by bisection, the bases FB, FD are equal; and the angle CBF equals the angle CDF (*a*): but CDF is the half of CDE; therefore CBF is the half of CBA: because CBA equals CDE. Therefore ABC is bisected by the straight line BF. In the same way, it may be shown, that the angles BAE, AED are bisected by the straight lines FA, FE.

From the point F draw FG, FH, FK, FL, FM, perpendicular upon the sides of the pentagon (*b*). Now, in the triangles FCH, FCK, the side FC is common; and the angles at C are equal, by bisection; and those at H, K are equal, as right angles (*c*); therefore the sides FH, FK are equal (*d*). In like manner, it may be shown, that each of the perpendiculars FL, FM, FG is equal to FH, or FK: therefore, these five straight lines are equal to one another; and a circle described upon the centre F, at the distance of any one of them, will pass through the extreme points of the other four; and touch the sides of the pentagon, where they meet the perpendiculars, in the points G, H, K, L, M (*e*). But a circle is inscribed in a rectilineal figure, when the circumference touches all the sides of the figure (*f*): wherefore a circle is inscribed, &c., which was to be done.

Recite (*a*) p. 4, 1; (*b*) p. 12, 1; (*c*) ax. 10; (*d*) p. 26, 1; (*e*) p. 16 and cor. 3; (*f*) def. 4, 4.

**14 P.** To describe a circle about a given equilateral and equiangular pentagon (ABCDE).

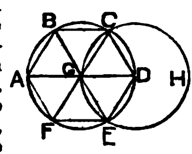



Bisect two of the angles of the pentagon, as BCD, CDE, by straight lines CF, DF, meeting in the point F (*a*): join FB, FA, FE.

Argument. Because the equal angles BCD, CDE are bisected; the angles FCB, FCD are equal; and they are contained by the equal sides CB, CF; CD, CF: therefore the remaining sides FB, FD are equal, and also the angle CDF to CBF (*b*): but CDF is the half of CDE, which equals CBA; therefore FB bisects the angle CBA. In like manner it may be shown, that the angles BAE, AED are bisected by the straight lines FA, FE. Now the five triangles, whose vertices are in the point F, have equal bases; namely, the sides of the pentagon; also equal angles adjacent to the bases (*c*), as above; therefore the sides FA, FB, FC, FD, FE, are equal to one another; and being drawn from the point F to the angular points of the pentagon, a circle described upon F, at the distance of any one of them, will pass through the five points, and be described about the pentagon; which was to be done (*d*).

Recite (*a*) p. 9, 1; (*b*) p. 4, 1;
(*c*) p. 6, 1; (*d*) def. 3, 4.

**15 P.** To inscribe an equilateral and equiangular hexagon in a given circle (ABCDEF).




Let G be the centre of the given circle, and draw the diameter AGD (*a*): Again, on the point D, where the diameter meets the circumference, describe a circle to pass through G, and cut the circumference in the points E, C (*b*): draw the diameters EGB, CGF: also chords between the points A, B, C, D, E, F, A, in the circumference. If these chords be equal, and their angles equal, the required hexagon is inscribed.

Because G and D are centres of equal circles, the radii GE, GD and DE are equal (*c*): GC, GD and DC are in the same case: therefore, the triangle GED, or GCD is equilateral (*d*); and being every way isosceles, it is also equiangular (*e*): therefore, the angle CGD is one third of two right angles (*f*), EGD is also one third of the same; and because the straight line EG makes with CF, the adjacent angles equal to two right angles (*g*), therefore EGF is also one third of two right angles. The chord EF is therefore equal to ED, or DC (*h*); and these three are placed in the semicircle (*i*). The opposite, or vertical angles are also equal to these (*k*): therefore each of the angles AGF, AGB, BGC, is one third of two right angles (*f*); and the radii GB, GA, GF, being equal (*c*), the chords AF, AB, BC are also equal to one another; and they are placed in a semicircle (*i*). Therefore the six chords AB, BC, CD, DE, EF, FA, divide the circumfer-

ence, cut off equal arcs (*l*), and are therefore sides of an equilateral hexagon inscribed in the given circle ABCDEF.

But, since equal angles stand upon equal arcs (*m*), the arc AF equals the arc ED; to both add the arc ABCD (*n*): therefore the whole arc FABCD equals the whole ABCDE; and the angle FED stands upon the former, and the angle AFE upon the latter: therefore the angles AFE, FED are equal. In the same way, it may be proved, that each of the other angles of the hexagon is equal to AFE, or FED. The hexagon (*o*) is therefore equiangular, and it was shown to be equilateral; and it is inscribed in the given circle: which was to be done.

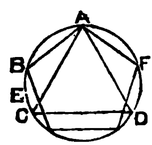
*Recite* (*a*) def. 16, 1; p. 15, 3; (*b*) p. 10, 3;
(*c*) def. 15, 1; (*d*) def. 23, 1; (*e*) def. 24, p. 5, 1;
(*f*) p. 32, 1; (*g*) p. 13, 1; (*h*) p. 4, 1;
(*i*) p. 1, 4; (*k*) p. 15, 1; (*l*) p. 28, 3;
(*m*) p. 26, 29, 3; (*n*) ax. 2; (*o*) Note to def. 36, 1.

Cor. The side of the hexagon; that is, the chord of one sixth part of the circumference, is equal to the radius, or semi-diameter of the circle.

Scholium. To describe an equilateral and equiangular hexagon about the circle and about the inscribed hexagon is the same thing: for if through the points A, B, C, D, E, F, tangents be drawn, touching the circle at right angles to the diameters, the angles of the inscribed hexagon will be in the sides of the one described. Def. 2, 4, b. 4.

*Tangent* literally means *they touch*. See def. 2, b. 3.

## 16 P. To inscribe an equilateral and equiangular quindecagon in a given circle (ABCD).




Let AC be the side of an equilateral triangle (*a*), and AB the side of an equilateral pentagon (*b*) inscribed in the given circle. Therefore, as it is required to cut the circumference into fifteen equal parts, the chord AC cuts off five, AB cuts off three, and the difference BC contains two of those fifteenths. Bisect the arc BC in E (*c*): therefore, the arc BE, or EC, is one fifteenth part of the circumference. Now, if the chord BE, or EC be drawn, and equal chords be placed all around in the circle (*d*), an equilateral and equiangular quindecagon shall be inscribed in it; which was to be done.

And, if through the angular points of the inscribed quindecagon, tangents be drawn, an equilateral and equiangular quindecagon shall be described about the circle, and also about the inscribed quindecagon: for the angular points of the inscribed rectilineal figure shall be in the sides of the one described (*e*).

Recite (*a*) p. 2, 4; (*b*) p. 11, 4; (*c*) p. 30, 3; (*d*) p. 1, 4;
(*e*) def. 36, 1, and Note; def. 1, 2, 3, 4, 5, of b. 1.

# BOOK FIFTH.

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## *Definitions.*

1. A less magnitude is said to be a *part* of a greater one, when the less is a measure of the greater, or is contained a certain number of times in it.

2. A greater magnitude is said to be a *multiple* of a less one, when the greater is measured by the less, or contains it a certain number of times.

A. There is a series of multiples; as, the first, second, third, &c., of which, waving the etymology of the word, the magnitude itself is the first, its double is the second, its triple the third, &c.

B. A magnitude may have one, two, or three dimensions, as the case may be; and the proper unit of measure will be a line, square surface, or cube.

3. Ratio is the numerical relation of antecedent and consequent, or the number of times, or parts of times, which the latter contains the former.

Or, Ratio is the numerical relation of measure and magnitude, or the number of times which the measure, or a part of it, may be applied to the magnitude.

*Note.*—This value of ratio prevails; and the words of several propositions are here changed to correspond with it.

4. Magnitudes of the same kind only, or having some common property, can have a ratio to one another.

5. The first of four magnitudes has the same ratio to the second which the third has to the fourth, when equimultiples of the first and third, also of the second and fourth, being taken; if the multiple of the first be greater than that of the second, the multiple of the third is greater than that of the fourth; if equal, equal; and if less, less.

6. Magnitudes which have the same ratio are called proportionals; of which it is usually said, "the first *is to* the second *as* the second *is to* the third; or, the first *is to* the second *as* the third *is to* the fourth.

*Note.*—The *is to*, as above, is expressed by a colon, thus, (:), the *as* by two colons, thus (: :).

7. The ratio of one couplet (or antecedent and consequent) is less than the ratio of another couplet, when the quotient of the former consequent, divided by its antecedent, is less than the quotient of the latter consequent divided by its antecedent.

8. When three terms or magnitudes are proportionals, the ratio of the first to the third is the duplicate, or square of the ratio of the first to the second.

9 When four terms are continued proportionals; that is, when

the second is the consequent of the first, the third that of the second, and the fourth that of the third; then the ratio of the first to the fourth is the triplicate or cube of the ratio of the first to the second. Such ratios are called compound.

10. And when any number of magnitudes of the same kind are in a certain order, however different the ratios of the couplets may be, the ratio of the first to the last of them is the continual product of all the ratios; namely, the product of all the antecedents for an antecedent, and the product of all the consequents for a consequent.

11. In proportionals, taken, two and two, from different series, or from remote terms of the same series, the odd terms, namely, the first, third, fifth, &c., are the antecedents; and these are said to be homologous; so, likewise, the even terms, viz. the second, fourth, sixth, &c., which are the consequents.

Geometers use the terms *permutando*, or *alternando*, *invertendo*, *componendo*, *dividendo*, *convertendo*, *ex æquali distantia*, *ex æquo*, and *ex æquali*, *in proportione perturbata*, *vel inordinata*, to signify various changes in the order, or magnitude of proportionals, and still preserving the equality of the ratios, in which proportion consists. The sense of these terms is expressed in the following examples:

The use of the marks +, —, ×, ÷, :, : :, and =, is generally known.

Example 1. By permutation, or alternately; when, of four proportionals, as A : B : : C : D, comes A : C : : B : D. See p. 16 of b. 5.

Ex. 2. By inversion; when, of four proportionals, as A : B : : C : D, comes B : A : : D : C. See p. B. of b. 5.

Ex 3. By composition; when, of four proportionals, as A : B : : C : D, comes A + B : B : : C + D : D. See p. 18 of b. 5.

Ex. 4. By division; when, of four proportionals, as A : B : : C : D, comes A — B : B : : C — D : D. See p. 17 of b. 5.

Ex. 5. By conversion; when, of four proportionals, as A : B : : C : D, comes A : A — B : : C : C — D. See p. E of b. 5.

Ex. 6. From equal distance in order; when, of two ranks of proportionals, as A, B, C, D, and E, F, G, H, taken, two and two, in order, namely, A : B : : E : F;—B : C : : F : G; and C : D : : G : H;—it comes to be inferred, that A : D : : E : H. See p. 22 of b. 5.

Ex. 7. From equal distance out of order; when, of two ranks of proportionals, as A, B, C, D, and E, F, G, H, taken, two and two, in a cross order, namely, A : B : : G : H;—B : C : : F : G; and C : D : : E : F; it comes to be inferred, that A : D : : E : H. See p. 23 of b. 5.

## *Axioms.*

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

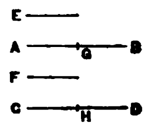
2. Those magnitudes are equal to one another, of which the same, or equal magnitudes, are equimultiples.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. One magnitude is greater than another, of which a multiple is greater than the same multiple of the other.

## *Propositions.*

1 Th. If two or more magnitudes be equimultiples of as many parts, each of each; what multiple soever any one of them is of its part, the same shall the sum of all the magnitudes be of the sum of all the parts.




Let AB, CD be two magnitudes, and E, F, two parts; so that AB be the second multiple of E, and CD the second multiple of F (*a*); AB+CD is the second multiple of E+F.

For since in AB there are two magnitudes AG, GB, each equal to E, and in CD there are two, CH, HD, each equal to F; then AG+CH=E+F, and GB+HD=E+F: therefore AG+CH+GB+HD=AB+CD=2(E+F), (*b*).

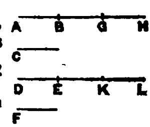
In like manner, if AB, CD were third multiples of E, F, their sum would be third multiples of the sum of E and F; and so of any equimultiples whatever.

Also, if there were three, four, or more magnitudes, equimultiples of as many parts; the sum of all the magnitudes would be the same multiple of the sum of all the parts that each magnitude would be of its part.

Wherefore, if two, or more, &c. Q. E. D.

Recite (*a*) def. 1, 2, A of b 5; (*b*) ax. 2 of b 1.

2 Th. If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then the sum of the first and fifth, and of the third and sixth are equimultiples of the second and fourth respectively.



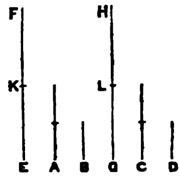

Let AB, C, DE, F, BH, EL be six magnitudes, in order: and since multiples are a numerical series of magnitudes (*a*); let AB=1C, and DE=1F; also, let BH=2C, and EL=2F: therefore the sum of AB, BH=3C, and the sum of DE, EL=3F, the same multiple of C and F.

Wherefore, if the first, &c. Q. E. D.

Recite (*a*) def. 1, 2, A of b 5.

Scholium. It advances no general principle to use letters instead of numbers in the case of equimultiples. There is no variety in the case: it is only necessary to understand that the multiples are the same in every set of magnitudes so compared; and there is no better way to express the same multiples than by the same numbers undisguised with symbols.

3 Th. If the first be the same multiple of the second which the third is of the fourth; and if of the first and third equimultiples be taken, these shall be equimultiples, one of the second the other of the fourth.




Let A, B, C, D be four magnitudes, in order; such that A is the second multiple of B, and C the second multiple of D; and if EF be taken the second multiple of A, and GH the second multiple of C: then will EF be the fourth multiple of B, and GH the fourth multiple of D (*a*).

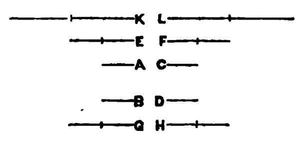
In like manner, if A be the third multiple of B, and C the third multiple of D; and if EF be taken the second multiple of A, and GH the second multiple of C; then EF is the sixth multiple of B, and GH the sixth multiple of D.

And, in general, because B measures A as often as D measures C; and A measures any multiple of itself as often as C measures the same multiple of itself: therefore B measures any multiple of A, as often as D measures the same multiple of C (*a*).

Wherefore, if the first be the same multiple, &c. Q. E. D.

Recite (*a*) definitions 1, 2, A of b 5

4 Th. If the first of four magnitudes have the same ratio to the second which the third has to the fourth; then any equimultiples of the antecedents shall have the same ratio as any equimultiples of the consequents.




Let A, B, C, D, be four magnitudes; such that A is to B as C is to D; of which A and C are the antecedents, and B and D the consequents (*a*). Take E, F equimultiples of A, C, and G, H equimultiples of B, D. It is inferred that E is to F as G is to H (*b*).

Take K, L equimultiples of E, F, and M, N equimultiples of G, H. Then, because E, F are equimultiples of A, C, and K, L of E, F, the same K, L are equimultiples of A, C (*c*). Likewise, because G, H are equimultiples of B, D, and M, N of G, H, the same M, N are equimultiples of B, D (*c*).

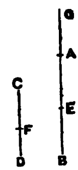
And since A is to B as C is to D, by hypothesis, K is to L as M is to N. Therefore, if K be greater than M, L is greater than N; if equal, equal; and if less, less (*b*). And K, L are equimultiples of E, F, and M, N of G, H: therefore E is to F as G is to H.

Wherefore, if the first of four, &c. Q. E. D.

Cor. And, if A is to B as C is to D; that is, if 1A is to 1B as 1C is to 1D, then 2A is to B as 2C is to D; or A is to 2B as C is to 2D; and so of any equimultiples whatever.

Recite (*a*) def. 3, 5; (*b*) def. 5, 5; (*c*) p. 3, 5;

5 Th. If one magnitude be the same multiple of another, that a part of the one is of a part of the other, the remainder shall be the same multiple of the remainder that the whole is of the whole.



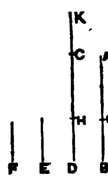

Let AB be the same multiple of CD that the part AE is of the part CF; the other part EB is the same multiple of the part FD that AB is of CD.

Make AG the same multiple of FD that AE is of CF; then AE is to CF as EG is to CD, or AB to CD: hence AB and EG are equimultiples of CD, and are equal to each other (*a*). Take AE from both; then AG and EB are left equal (*b*). But AG is to FD as AE is to CF, or as AB is to CD: therefore EB is to FD as AB is to CD; or, in other words, EB is the same multiple of FD that AB is of CD.

Q. E. D.

Recite (*a*) p. 1, and ax. 1, 5; (*b*) ax. 3, 1.

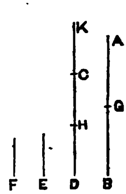
6 Th. If two magnitudes be equimultiples of *other two*; and if equimultiples of the latter be taken from the former; the parts left, if any, will be equals, or equimultiples of the *other two*.




Let the two magnitudes AB, CD, be equimultiples of the two, E, F; and the parts AG, CH equimultiples of the same E, F; the remainders GB, HD, are equal to E, F, or they are equimultiples of them.

1. If GB equal E, HD will equal F: make CK=F. Then, since AG, CH are equimultiples of E, F; and GB=E, and CK=F; therefore AB, KH are equimultiples of E, F; but AB and CD are equimultiples of E, F; therefore KH equals CD (*a*). Take CH from both; the remainders KC, HD are equal: but KC=F.

Therefore HD is equal to F

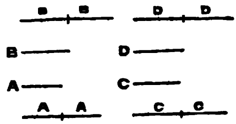



2. If GB be a multiple of E, HD is the same multiple of F. Make CK the same multiple of F that GB is of E: then since AG, CH are equimultiples of E, F; and GB, CK equimultiples of E, F; AB, KH are also equimultiples of E, F (*b*): therefore KH, CD are the same multiple of F, and are equal (*a*) Take CH from both; the remainders KC, HD are equal. And since GB, KC are equimultiples of E, F, and KC=HD; therefore GB, HD are equimultiples of E, F.

Wherefore, if two magnitudes, &c. Q. E. D.

Recite (*a*) ax. 1, 5; (*b*) p. 2, 5.

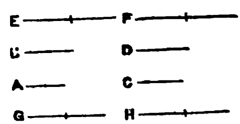
A Th. If the first of four magnitudes have to the second the same ratio which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.




Let equimultiples be taken of the four magnitudes A, B, C, D: and because A is to B as C is to D, 2A is to 2B as 2C is to 2D: therefore, if 2A exceed 2B, 2C will exceed 2D. But if A be greater than B, 2A must be greater than 2B; and so, 2C is greater than 2D; but if 2C exceed 2D, C is greater than D. In like manner, if the first equal the second, or is less than it, it may be proved that the third equals the fourth, or is less than it.

Therefore, if the first of four, &c. Q. E. D.

B Th. If four magnitudes be proportionals, they are proportionals also when taken inversely.



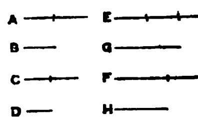

If A is to B as C is to D; then, inversely, B is to A as D is to C. The antecedents are A, C, the consequents B, D.

Take E, F, equimultiples of the consequents, and G, H, equimultiples of the antecedents. Now, because B contains A as often as D contains C (*a*), and E, F are equimultiples of B, D; therefore E contains A as often as F contains C (*a*). But G, H are equimultiples of A, C: therefore, if E be greater than G, F is greater than H; if equal, equal: and if less, less (*b*). But E, F are equimultiples of B, D, and G, H of A, C: therefore B is to A as D is to C.

Therefore, if four magnitudes, &c. Q. E. D.

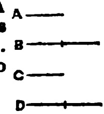
Recite (*a*) def. 3, 5; (*b*) def. 5, 5.

C Th. If the first be the same multiple, or part of the second, that the third is of the fourth, the first is to the second as the third is to the fourth.




Let A, B, C, D be the magnitudes, in order. And, first, let A, C be equimultiples of B, D; then A is to B as C is to D.

Take E, F equimultiples of A, C, and G, H equimultiples of B, D. Then, because A, C are equimultiples of B, D (*a*); and E, F equimultiples of A, C; therefore E, F are equimultiples of B, D (*b*). But G, H are equimultiples of B, D, any whatever: therefore, if E exceeds G, F will exceed H; or, if E be equal to G, or less than it, F is equal to H, or less than it. But E, F are equimultiples of the first and third; and G, H of the second and fourth: therefore A is to B as C is to D (*c*).



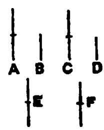

Next, let A, B, C, D be the terms in order; and let A and C be the same parts of B and D: A is to B as C is to D. For B is the same multiple of A that D is of C. Wherefore, by the preceding case, B is to A as D is to C; and inversely A is to B as C is to D (*d*).

Therefore, if the first be the same, &c.

Q. E. D.

Recite (*a*) the hypothesis; (*b*) p. 3, 5; (*c*) def. 5, 5; (*d*) p. B, 5.

D Th. If the first be to the second as the third is to the fourth, and if the first be a multiple or part of the second, the third is the same multiple or part of the fourth.




Let A, B, C, D be four magnitudes, in order; so that A is to B as C is to D.

First, if A, C be equimultiples of B, D, take E equal to A: then make F the same multiple of D that A, or E is of B. And because A is to B as C is to D; and E, F are taken equimultiples of B, D; therefore A is to E as C is to F (*a*). But A equals E, therefore C equals F (*b*); and F, A are equimultiples of D, B: wherefore C is the same multiple of D that A is of B.

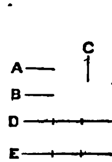
Again, see the figure, second case of p. C, above.

If A, C be equal parts of B, D; because A is to B as C is to D; then, inversely, B is to A as D is to C (*c*). But A is a part of B, and so B is a multiple of A (*d*); but by the preceding case, D, B are equimultiples of C, A: therefore C is the same part of D that A is of B.

Hence, if the first, &c. Q. E. D.

Recite (*a*) cor. p. 4, 5; (*b*) p. A, 5; (*c*) p. B, 5; (*d*) def. 1 and 2, b. 5.

7 Th. Equal magnitudes have the same ratio to a magnitude; and a magnitude has the same ratio to equal magnitudes.




The magnitudes A, B, being equal, have to C the same ratio: and the magnitude C has to A and B the same ratio.

Take D, E any equimultiples of A, B, and F any multiple of C: then because D, E are equimultiples of the equals A, B, they are equal to one another (*a*). Therefore if D be greater than F, E is greater than F; if equal, equal; and if less, less (*b*). And D, E, are any equimultiples of A, B, and F is any multiple of C: therefore A is to C as B is to C.

C has also the same ratio to A that it has to B: for D may be shown equal to E, as before: and, if F be greater than D, it is also greater

than E; if equal, equal; and if less, less. And F is any multiple of C, and D, E are any equimultiples of A, B. Therefore C is to A as C is to B (*b*).

Therefore equal magnitudes, &c. Q. E. D.

Recite (*a*) ax. 1, 5; (*b*) def. 5, 5.

**8 Th.** Of unequal magnitudes the greater has a less ratio to the same than the less has to it; and a magnitude has a less ratio to the less than to the greater.

Given A, B, unequal magnitudes of which A is the less; also C, a magnitude of the same kind as A and B (*a*); so that the same measuring unit (*b*), may apply to A, B, C.


1. The ratio of B to C is less than that of A to C. Because B is greater than A, it is also a greater multiple of their common measuring unit; and C will contain the greater of two multiples a less number of times than it will contain the other: but C is the consequent, and A and B are its antecedents (*c*); therefore the ratio of B to C is less than that of A to C (*d*).

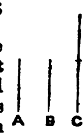
2. The ratio of C to A is less than that of C to B. Let some measure be applied to C, which will also measure A and B: then because A is less than B, it will contain the measure applied to C, or any multiple of it, a less number of times than B will contain it: but C is a common antecedent, and A and B are its consequents (*d*); therefore, the ratio of C to A is less than that of C to B.

Wherefore, of unequal magnitudes, &c. Q. E. D.

Recite (*a*) def. A, and 4, 5; (*b*) def. 2, 1, and B, 5; (*c*) def. 3, 5: (*d*) def. 7, 5.

**9 Th.** Magnitudes are équal to each other which have the same ratio to a magnitude; and those are equal magnitudes to which a magnitude has the same ratio.

1. Given the ratio of A to C the same as that of B to C; A is equal to B.



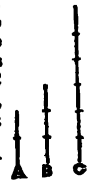

Because A and B have a ratio to C, the magnitudes are of the same kind (*a*), and are measured by the same unit (*b*): for the same reason A and B are the antecedents and C the consequent of the ratios (*c*): and because the ratios are equal, A and B are equimultiples of their common measure (*d*); therefore A and B are equal magnitudes.

2. Given the ratio of C to A equal to that of C to B. A, B, and C are magnitudes of the same kind, and measured by the same unit, as above; and because C has a ratio to A and B, C is the antecedent and A and B are the consequents of the ratios (*c*); and because the ratios are equal, A and B are equimultiples of their common measure (*d*): therefore A and B are equal magnitudes.

Wherefore, magnitudes are equal, &c. Q. E. D.

Recite (*a*) def. 4, 5; (*b*) def. B, 5;
(*c*) def. 3 and 7, 5; (*d*) def. A, 5.

**10 Th.** That is the less magnitude of two, which has a greater ratio to a third magnitude; and that magnitude is the greater of two, to which a third magnitude has a greater ratio.



1. Given A to C greater than B to C; A is less than B. Because A and B have ratios to C, the magnitudes are of the same kind (*a*), and are measured by the same unit (*b*); for the same reason A and B are the antecedents and C is the consequent of the ratios (*c*); and because C contains A a greater number of times than it contains B, A is a less multiple of the common measure than B is: therefore A is a less magnitude than B.

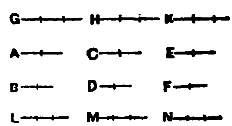
2. Given C to B greater than C to A; B is greater than A.

A, B and C are magnitudes of the same kind, and measured by the same unit, as above; and because C has ratios to A and B, C is the antecedent, and A and B are the consequents of the ratios (*c*); and because B contains the unit of measure applied to C, or a multiple of it, a greater number of times than A contains the same unit, or the same multiple of it; therefore B is a greater magnitude than A.

Wherefore, that is the less magnitude, &c. Q. E. D.

Recite (*a*) def. 4, 5; (*b*) def. B, 5; (*c*) def. 3 and 7, 5.

**11 Th.** Ratios that are the same to the same ratio are the same to one another.



Given A to B as C to D,
and C to D as E to F:
then, A is to B as E is to F.

The antecedents are A, C, E;
the consequents, B, D, F.

Take G, H, K equimultiples of A, C, E; and L, M, N equimultiples of B, D, F (*a*).

Then, since A is to B as C is to D; and G, H are equimultiples of A, C, and L, M of B, D; therefore, if G be greater than L, H is greater than M; if equal, equal; and if less, less.

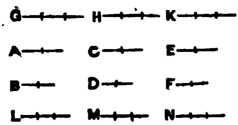
Again, since C is to D as E is to F; and H, K are equimultiples of C, E and M, N of D, F; therefore, if H be greater than M, K is greater than N; if equal, equal; and if less, less.

But it proves, as above, that if G be greater than L, H is greater than M; if equal, equal; and if less, less: therefore, if G be greater than L, K is greater than N; if equal, equal; and if less, less. And G, K are equimultiples of A, E; and L, N of B, F. Therefore A is to B as E is to F (*b*).

Wherefore, ratios that are the same, &c. Q. E. D.

Recite (*a*) def. 5, 5; (*b*) ax. 1, 1.

**12 Th.** If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall the sum of all the antecedents be to the sum of all the consequents.




Given the magnitudes A, B; C, D; E, F; so that A is to B as C is to D, or as E is to F: then A is to B as A+C+E is to B+D+F.

Take G, H, K equimultiples of A, C, E; and L, M, N equimultiples of B, D, F.

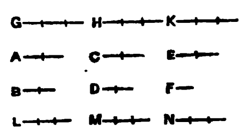
From this arrangement, if G be greater than L, H is greater than M, and K greater than N; if equal, equal: and if less, less (*a*). Wherefore, if G be greater than L, G+H+K is greater than L+M+N; if equal, equal; and if less, less. But G and G+H+K are equimultiples of A and A+C+E (*b*); also L and L+M+N are equimultiples of B and B+D+F (*b*).

Therefore, A is to B as A+C+E is to B+D+F.

Wherefore, if any number of magnitudes, &c. Q. E. D.

Recite (*a*) def. 5, 5; (*b*) p. 1, 5.

**13 Th.** If the first have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall have to the second a greater ratio than the fifth has to the sixth.




Take A, B; C, D; E, F, six magnitudes, two and two in order, of the same kind (*a*).

Then, because A has a ratio to B, C to D, and E to F, the same unit will measure the antecedent and consequent of each couplet (*b*).

And because A is to B as C is to D, B contains A as many times, or parts of times, as D contains C; and because C is to D greater than E is to F, D contains C more times, or parts of times, than F contains E (*c*): but the quotients of B divided by A, and of D divided by C prove equal; therefore the quotient of B by A is greater than that of F by E (*d*)—that is, the ratio of A to B is greater than the ratio of E to F.

Wherefore, if the first has to the second, &c. Q. E. D.

Recite (*a*) def. 4, 5; (*b*) def. B, 5;
(*c*) def. 3, 5; (*d*) def. 7, 5.

Cor. If the order of the couplets were transposed; it might be demonstrated, in the same way, that the ratio of the first to the second is less than that of the fifth to the sixth.

**14 Th. If the first have to the second the same ratio which the third has to the fourth; then if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.**

Given A to B as C to D, and A, C antecedents, B, D consequents.

1. If A be greater than C, the quotient of B divided by A will be less than that of B divided by C (*a*); but the quotients of B by A and D by C are given equal; for the consequent divided by the antecedent is the ratio (*b*): therefore the quotient of B divided by C is greater than the quotient of D divided by C; and so, B is greater than D.

2. If A equals C, the two quotients of B divided by A and by C are equal (*c*); but the quotients of B by A and D by C are given equal, as above (*b*): therefore the quotients of B and D by C are equal: and so, B is equal to D.

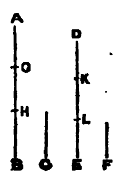
3. If A be less than C, the quotient of B by A is greater than that of B by C (*a*): but the quotients of B by A aud D by C are given equal (*b*): therefore the quotient of B by C is less than that of D by C; and so, B is less than D.

Wherefore, if the first have to the second, &c. Q. E. D.

Recite (*a*) p. 8, 13, 5; (*b*) def. 3, 5; (*c*) p. 9, 5.

NOTE.—The ratio of C to B is here introduced as a medium of comparison between B and D; also the magnitudes are so divided in the diagram that each may be taken greater or less as the case may require.

**15 Th. Magnitudes have the same ratio to one another which their equimultiples have.**



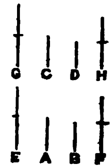
Given two magnitudes C, F; and equimultiples of them, AB, DE: C is to F as AB is to DE.

Because AB is a multiple of C, C is a part of AB (*a*): for the same reason F is a part of DE. And because AB, DE are equimultiples of C, F; AB contains the measure C as often as DE contains the measure F (*b*). Apply the measure C, from A to G, from G to H, and from H to B: apply also the measure F, from D to K, from K to L, and from L to E. Then, because the parts AG, GH, HB, are equal; and the parts DK, KL, LE are equal; AG is to DK as GH is to KL, as HB is to LE (*c*). Now AG, GH, HB are the antecedents, and DK, KL, LE are the consequents; wherefore AG is to DK as AB is to DE (*d*): but AG is equal to C and DK to F; therefore, C is to F as AB is to DE.

Therefore, magnitudes have the same ratio, &c. Q. E. D.

Recite (*a*) def. 1, 2 of 5; (*b*) def. A, 5;
(*c*) p. 14, 5 (case 2); (*d*) p. 12, 5.

**16 Th. If four magnitudes of the same kind be proportionals, they shall be proportionals also when taken alternately.**




If A, B, C, D be four magnitudes of the same kind, and have A to B as C to D; then, alternately, A : C : : B : D.

Take E, F equimultiples of A, B, and G, H equimultiples of C, D.

Because E, F are equimultiples of A, B, and that magnitudes have the same ratio which their equimultiples have (*a*); therefore A is to B as E is to F: but A is to B as C is to D (*b*); therefore C is to D as E is to F.

Again, because G, H are equimultiples of C, D; therefore C is to D as G is to H (*a*): but C is to D as E is to F, as above; therefore E is to F as G is to H (*c*); and so, the equimultiples are proportionals (*d*). Wherefore, if E be greater than G, F is greater than H; if equal, equal; and if less, less (*e*). But E, F are equimultiples of A, B, and G, H of C, D: therefore A is to C as B is to D.

Wherefore, if four magnitudes of the same kind, &c.

Q. E. D.

Recite (*a*) p. 15, 5; (*b*) hypothesis; (*c*) p. 11, 5;
(*d*) def. 6, 5; (*e*) p. 14, 5.

**17 Th. If the sum of two magnitudes have to one of them the same ratio which the sum of other two has to one of these, the one left of the former two shall have to the other the same ratio which the one left of the latter two has to the other of these.**

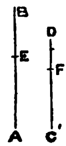

Let AE, EB be two magnitudes, and CF, FD other two: the sum of the former is AE+EB, of the latter CF+FD.

Then, since AE+EB : EB : : CF+FD : FD (*a*); inversely, EB : AE+EB : : FD : CF+FD (*b*). And since ratio is the quotient of the consequent divided by the antecedent (*c*); and division is indicated by writing the divisor under the dividend: $\frac{AE+EB}{EB}=\frac{CF+FD}{FD}$; that is, $\frac{AE}{EB}+\frac{EB}{EB}=\frac{CF}{FD}+\frac{FD}{FD}$. But $\frac{EB}{EB}=\frac{FD}{FD}$; take these equals from the former (*d*); there will remain $\frac{AE}{EB}=\frac{CF}{FD}$. Now draw these equals into line; therefore, EB : AE : : FD : CF; and inversely, AE : EB : : CF : FD (*b*).

Wherefore, if the sum of two magnitudes, &c. Q. E. D.

Recite (*a*) hyp. (*b*) p. B, 5; (*c*) def. 3, 5;
(*d*) ax. 3, 1.

18 Th. If the first be to the second as the third is to the fourth, the sum of the first and second shall be to the second, as the sum of the third and fourth is to the fourth.



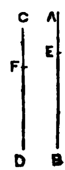

Given AE to EB as CF to FD, four magnitudes: then AE+EB : EB : : CF+FD : FD.

Because AE is to EB as CF is to FD, by hyp. inversely, EB : AE : : FD : CF (*a*); and because ratio is the quotient of the consequent by the antecedent (*b*), $\frac{AE}{EB}=\frac{CF}{FD}$. Again, $\frac{EB}{EB}=\frac{FD}{FD}$: add these equals to the former; then $\frac{AE}{EB}+\frac{EB}{EB}=\frac{CF}{FD}+\frac{FD}{FD}$ (*c*); that is, $\frac{AE+EB}{EB}=\frac{CF+FD}{FD}$ Now draw these equals into line; therefore EB : AE+EB : : FD : CF+FD; and inversely, (*a*) AE+EB : EB : : CF+FD : FD, as stated.

Wherefore, if the first be to the second, &c. Q. E. D.

Recite (*a*) p. B, 5; (*b*) def. 3, 5; (*c*) ax. 2, 1.

19 Th. If one magnitude be to another as a part of the one is to a part of the other, the parts left have the same ratio as the whole magnitudes.



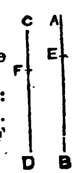

Let the two magnitudes AB, CD have the same ratio as the parts AE, CF; then the other parts EB, FD have the same ratio as AB to CD.

Because AB=AE+EB, and CD=CF+FD: therefore AE+EB : CF+FD : : AE : CF (*a*); and, alternately, AE+EB : AE : : CF+FD : CF (*b*): but these are joint proportionals, which may be taken separately: therefore AE : EB : : CF : FD (*c*); and, alternately, AE : CF : : EB : FD (*b*). But AE is to CF as AB is to CD (*a*); therefore, also, EB is to FD as AB is to CD (*d*).

Wherefore, if one magnitude be, &c. Q. E. D.

Recite (*a*) hyp.; (*b*) p. 16, 5;
(*c*) p. 17, 5; (*d*) p. 11, 5.

E Th. Of four proportionals, the first is to its excess above the second as the third is to its excess above the fourth.




Given AB : BE : : CD : DF; then, by hyp.,
AB : AB—BE : : CD : CD—DF.

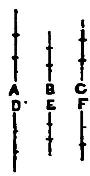
Let the parts of AB be AE, EB, and the parts of CD be CF, FD.

Then, since AB : EB : : CD : FD; by division (*a*) AE : EB : : CF : FD; and by inversion EB : AE : : FD : CF (*b*). Wherefore, by composition. AE+EB : AE : : CF+FD : CF (*c*): that is, AB : AB—BE : : CD : CD—DF, as stated.

Wherefore, of four proportionals, &c. Q. E. D.

Recite (*a*) p. 17, 5; (*b*) p. B, 5; (*c*) p. 18, 5.

**20 Th.** If there be three magnitudes, and other three, which, taken two and two, in order, have the same ratio: if the first be greater than the third, the fourth will be greater than the sixth; if equal, equal; and if less, less.




Take A, B, C and D, E, F, six magnitudes: then, by hypothesis, A : B : : D : E,
and B : C : : E : F.

Wherefore, since, in compound ratios, the product of the antecedents are antecedents, and the product of the consequents are consequents (*a*);

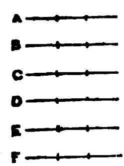
$A \times B : B \times C :: D \times E : E \times F.$

But since magnitudes have the same ratio as their equimultiples (*b*), the co-factors B, E may be rejected (*c*). Therefore A : C : : D : F. In which case, if A be greater than C, D will be greater than F; if equal, equal; and if less, less (*d*).

Wherefore, if there be three magnitudes, &c. Q. E. D.

Recite (*a*) def. 10, 5; (*b*) p. 15, 5;
(*c*) Note 3, page 32; (*d*) p. A of b. 5.

**21 Th.** If there be three magnitudes, and other three, which, taken two and two, out of order, have the same ratio; if the first be greater than the third, the fourth will be greater than the sixth; if equal, equal; and if less, less.




Take A, B, C and D, E, F, six magnitudes: then, by hypothesis, A : B : : E : F,
and B : C : : D : E.

Wherefore, since in compound ratios, the product of the antecedents are antecedents, and the product of the consequents are consequents (*a*);

$A \times B : B \times C :: D \times E : E \times F.$

But, since magnitudes have the same ratio as their equimultiples (*b*), the co-factors B, E may be rejected (*c*).

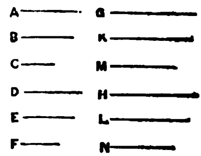
Therefore A : C : : D : F. In which case, if A be greater than C, D will be greater than F; if equal, equal; and if less, less (*d*).

Wherefore, if there be three magnitudes, &c. Q. E. D.

Recite (*a*) def. 10, 5; (*b*) p. 15, 5;
(*c*) Note 3, page 32; (*d*) p. A of b. 5.

NOTE.—The lines are so divided that the magnitudes A, B, C and D, E, F may be taken greater, equal, or less, as the case may require.

22 Th. If there be any number of magnitudes, and as many others; which, taken two and two in order, have the same ratio: then the first is to the last of one rank, as the first is to the last of the other rank.




Take the magnitudes A, B, C, D and E, F, G, H, in such wise, that A : B : : E : F
B : C : : F : G
and C : D : : G : H.

Then, since, in compound ratios, the product of the antecedents are antecedents, and the product of the consequents are consequents (*a*);

Therefore A×B×C : B×C×D : : E×F×G : F×G×H.

And, since magnitudes have the same ratio as their equimultiples (*b*), the co-factors B, C and F, G may be rejected (*c*):

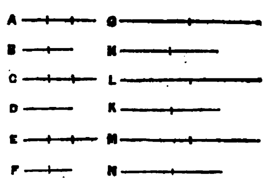
Therefore A : D : : E : H. In which case, if A be greater than D, E will be greater than H; if equal, equal; and if less, less (*d*).

Wherefore, if there be any number, &c. Q. E. D.

Recite (*a*) def. 10, 5; (*b*) p. 15, 5; (*c*) Note 3, page 32; (*d*) p. A of b 5.

N. B. This is cited *ex æquo*, simply; or, *ex æquali distantia*. In our example, page 70, it is given, "from equal distance in order;" that is, the homologous terms are equidistant in the order of the series of which the proportion is a part.

23 Th. If there be two ranks of magnitudes, which, taken two and two, out of order, have the same ratio; then, the first is to the last of one rank, as the first is to the last of the other.




Take the magnitudes A, B, C, D and E, F, G, H, in such wise, that
A : B : : G : H,
B : C : : F : G,
C : D : : E : F.

Then, since, in compound ratios, the products of the antecedents are antecedents, and the products of the consequents are consequents (*a*);

Therefore A×B×C : B×C×D : : E×F×G : F×G×H.

And since magnitudes have the same ratio as their equimultiples (*b*), the co-factors B, C and F, G may be rejected (*c*).

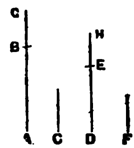
Therefore A : D : : E : H. In which case, if A be greater than D, E will be greater than H; if equal, equal; and if less, less (*d*).

Wherefore, if there be two ranks, &c. Q. E. D.

Recite (*a*) def. 10, 5; (*b*) p. 15, 5; (*c*) Note 3, page 32. (*d*) p. A of b 5.

N. B. This is cited *ex æquo perturbato*; or, *ex æquali distantia in proportione inordinata*. In our example, page 70, it is given, "from equal distance out of order," as exemplified above.

**24 Th.** If the first be to the second as the third is to the fourth, and the fifth to the second as the sixth to the fourth; then the sum of the first and fifth is to the second, as the sum of the third and sixth is to the fourth.




Take six magnitudes AB, C, DE, F, BG, EH, in order.

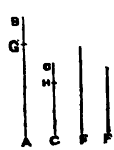
Then, by hyp. AB : C : : DE : F
and BG : C : : EH : F; and it is to be proved that AB+BG : C : : DE+EH : F.

Now, since BG : C : : EH : F
Inversely C : BG : : F : EH (*a*).
And since AB : C : : DE : F
Ex æquo AB : BG : : DE : EH (*b*). Again,
By composition AB+BG : BG : : DE+EH : EH (*c*).
Also, since BG : C : : EH : F, as above;
Ex æquo AB+BG : C : : DE+EH : F (*b*).
Wherefore, if the first be to the second, &c. Q. E. D.
Recite (*a*) p. B, 5; (*b*) p, 22, 5; (*c*) p. 18, 5.

Cor. 1. It may be proved, in the same way, that
AB—BG : C : : DE—EH : F. P. 19 of b 5.

Cor. 2. And in any two ranks of magnitudes, how many soever they may be in each rank, if each term of the first have to a magnitude the same ratio as each term of the second has to another magnitude, the sum of the first rank has to its magnitude the same ratio which the sum of the second rank has to its magnitude.

**25 Th.** If four magnitudes of the same kind be proportionals, the sum of the greatest and least of them is greater than the sum of the other two.




Let AB, CD, E, F be four proportionals, in order, namely, AB : CD : : E : F. And because they are of the same kind (*a*), they are measured by the same unit, and are terms of the same numerical series: therefore, the extremes in position are the extremes in magnitude.

Let AB be the greatest and F the least. In AB take AG=E, and in CD take CH=F. Then,
AB : CD : : AG : CH.
The remaining parts, GB, HD, have also the same ratio as the wholes (*b*); therefore AB : CD : : GB : HD. But AB is greater than CD; therefore GB is greater than HD (*c*).

And because AG=E, and CH=F; therefore AG+F=CH+E (*d*). To the left of this equation add GB, the greater; to the right add HD, the less: therefore AG+GB+F exceeds CH+HD+E (*e*); that is, AB+F is greater than CD+E.

Wherefore, if four magnitudes, &c. Q. E. D.
Recite (*a*) def. 4, 5; (*b*) p. 19, 5; (*c*) p. A, 5;
(*d*) ax. 2, 1; (*e*) ax. 4, 1.

F Th. Ratios composed of equal ratios are equal to each other.

Let the ratios of A to B and of B to C be severally equal to the ratios of D to E, and of E to F; the ratio of A to C equals that of D to F.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| For, let | A | to | B | equal | D | to | E, | |
| and | B | to | C | equal | E | to | F; | |
| then | A | is | to | C as | D | is | to | F—*ex æquo* (*a*). |
| Or, let | A | to | B | equal | E | to | F, | |
| and | B | to | C | equal | D | to | E; | |
| then | A | is | to | C as | D | is | to | F—*ex æquo perturbato* (*b*). |

Wherefore, ratios composed (or compounded), &c. Q. E. D.
Recite (*a*) p. 22, 5; (*b*) p. 23, 5.

G Th. If the same unit measure two magnitudes, it will also measure their sum; and if the magnitudes be unequal, it will measure their difference.

For since the same unit measures the two magnitudes, they are of the same kind, and the sum of their units may be placed in a consecutive series, to the whole of which the same measure applies. Again, if one be greater than the other, the difference is a multiple of the same measuring unit.

Therefore, if the same unit, &c. Q. E. D.

# BOOK SIXTH.

## *Definitions.*

1. Similar rectilineal figures are those which have their angles equal, each to each; and the sides about the equal angles proportionals.

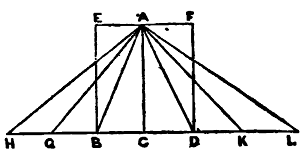
2. Reciprocal figures, viz. triangles and parallelograms, are those which have their sides about two of their angles proportionals, in such manner, that the extremes belong to one of the figures, and the means to the other; and such figures are equal to one another.

3 A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment to the less.

4. The altitude of any figure is the distance between two parallel lines touching the figure, one of which is its base.

## *Propositions.*

### 1 Th. Triangles and parallelograms of the same, or equal altitudes, are to each other as their bases




Given the triangles ABC, ACD, and the parallelograms CE, CF, on the bases BC, CD, and between two parallels (*a*); then ABC is to ACD, and CE is to CF, as BC is to CD.

Take HC, CL, equimultiples of BC, CD; and divide CH, CL into parts equal to CB, CD: then, because each of the parts GB, GH equals BC; and each of the parts KD, KL equals CD; therefore CH and CL are 3d multiples of BC and CD (*b*). To the point A join the points H, G, K, L: then the triangles ABC, AGB, AHG are all equal; and likwise the triangles ACD, ADK, AKL (*c*); and the triangles ACH, ACL are 3d multiples of the triangles ABC, ACD: and so, if the base CH be greater than the base CL, the triangle ACH is greater than the triangle ACL; if equal, equal; and if less, less. (*d*).
Now the bases BC, CD, and the triangles ABC, ACD, are four magnitudes; and of the 1st BC, and 3d ABC, equimultiples CH, ACH, are taken; and of the 2d and 4th CD, ACD, equimultiples CL, ACL, are taken; also it is shown, that if the base CH be greater than the base CL, the triangle ACH is greater than the triangle ACL; if equal,

equal; and if less, less: therefore, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD (*d*).

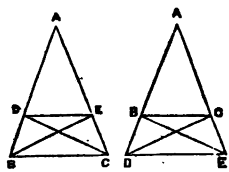
But the parallelograms CE, BF are 2d multiples of the triangles ABC, ACD (*e*), and the equimultiples have the ratio of their magnitudes (*f*); therefore, ABC : ACD : : CE : CF. And, since ABC : ACD : : BC : CD; therefore (*g*) also CE : CF : : BC : CD.

Wherefore, triangles, &c. Q. E. D.

Recite (*a*) def. 4, 6; (*b*) def. A, 5; (*c*) p. 38, 1;
(*d*) def. 5, 5; (*e*) p. 41, 1; (*f*) p. 15, 5;
(*g*) p. 11, 5.

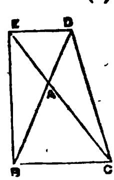
Cor. It may be also proved (in the same way,) that triangles and parallelograms of the same or equal bases are to each other as their altitudes.

2 Th. If a straight line be drawn parallel to one side of a triangle, it shall cut the other sides, or those sides produced proportionally: and if the sides, or the sides produced, be cut proportionally, the straight line joining the sectional points, shall be parallel to the third side of the triangle.




Given the triangle ABC, and DE drawn parallel to BC, cutting AB, AC; 1, within the triangle; 2, below the base; 3, above the vertex: in either case, BD : AD : : CE : AE. Join BE, CD.

1. The triangles BDE, CDE are equal; being on the same base DE, and between the same parallels BC, DE (*a*): but ADE is another triangle, to which the equals BDE, CDE have the same ratio (*b*); and BDE : ADE : : BD : AD (*c*); also, CDE : ADE : : CE : AE (*c*): therefore BD is to AD as CE is to AE (*d*).



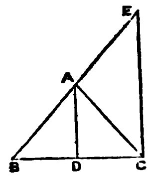

2. Upon the same construction. If BD : AD : : CE : AE, then DE is parallel to BC. For BD : AD : : BDE : ADE; and CE : AE : : CDE : ADE (*c*): therefore, BDE and CDE having the same ratio to ADE are equal (*b*); and being on the same base DE, and same side of it, they are also between the same parallels (*e*): therefore DE is parallel to BC.

Wherefore, if a straight line, &c. Q. E. D.

Recite (*a*) p. 37, 1; (*b*) p. 7, 9, of b. 5; (*c*) p. 1, 6;
(*d*) p. 11, 5; (*e*) p. 39, 1.

3 Th. If a straight line bisect an angle of a triangle, and also cut the base; the segments of the base shall have the same ratio as the sides which contain the angle: and

if the segments of the base have the same ratio as the other sides, the straight line drawn from the vertex shall bisect the vertical angle.




Given the triangle ABC, and AD bisecting the angle A: then BD : DC : : BA : AC;—and, if so, the angle A is bisected by AD.

Through C draw CE parallel to AD (*a*); and produce BA to meet CE in E.

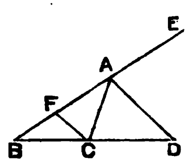
1. Because BE and AC meet the parallels AD, CE, the exterior angle BAD, and the interior BEC, or AEC are equal (*b*), also the alternate angles DAC, ACE (*c*): but BAD equals DAC, by hypothesis; therefore AEC equals ACE (*d*). and the side AE equals the side AC (*e*). And because AD is drawn parallel to CE; therefore, BD : DC : : BA : AE, (*f*), or its equal AC.

2. Because BD : DC : : BA : AC, by hyp. and, that from the parallels, BD : DC : : BA : AE (*f*); therefore AE equals AC (*g*); and their opposite angles AEC, ACE are equal (*h*): but AEC equals BAD (*b*), and DAC equals ACE; therefore BAD equals DAC (*i*), and so, the angle BAC is bisected by AD.

Wherefore, if a straight line bisect, &c. Q. E. D.

Recite (*a*) p. 31, 1; (*b*) p. 29, 1; (*c*) p. 27, 1;
(*d*) ax. 1, 1; (*e*) p. 6, 1; (*f*) p. 2, 6;
(*g*) p. 9, 5; (*h*) p. 5, 1; (*i*) ax. 1, 1.

A Th. If a straight line bisect the exterior angle of a triangle and meet the base produced, the segments between the meeting point and each extremity of the base shall have the same ratio as the other sides of the triangle. And if the segments of the base produced have the same ratio as the other sides, the straight line drawn from the vertex bisects the exterior angle.




Given the triangle ABC, its side BA produced to E, the exterior angle CAE bisected by AD which meets BC produced in D. BD : DC : : BA : AC; and if so, the exterior angle CAE is bisected by AD.

Through C draw CF parallel to AD (*a*).

1. Because the straight lines AC, FE meet the parallels AD, CF, the alternate angles DAC, ACF are equal (*b*), and so are the interior and exterior AFC, DAE (*c*): but by hyp. DAC equals DAE; therefore ACF equals AFC (*d*), and AF equals AC (*e*): and because CF is drawn parallel to AD; therefore BD : CD : : BA : AF (*f*), or AC its equal (*f*).

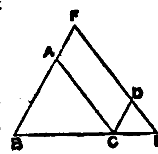
2. Again, because BD : DC : : BA : AC, or AF; therefore AC equals AF (*g*), and the angles AFC, ACF are equal (*h*); but the interior angle AFC equals the exterior DAE, and the alternate angles ACF, DAC are equal (*b*); therefore DAE equals DAC (*d*); and so, the angle CAE is bisected by AD.

Wherefore, if one side of a triangle be produced, &c.

Q. E. D.

Recite (*a*) p. 31, 1; (*b*) p. 27, 1; (*c*) p. 29, 1;
(*d*) ax. 1, 1; (*e*) p. 6, 1; (*f*) p. 2, 6;
(*g*) p. 9, 5; (*h*) p. 5, 1.

4 Th. The sides about the angles of equiangular triangles are proportionals; and the sides which are opposite to equal angles are homologous; that is, they are all antecedents, or all consequents of the ratios.




Given two triangles ABC, DCE, equiangular at B and C, at C and E, and therefore at A and D (*a*); the sides adjacent to these equal angles are proportionals, and the sides opposite to them are homologous.

Place the bases BC, CE in the same straight line; and because the angles ABC, ACB are less than two right angles (*b*), their equals ABC, DEC are also less, and BA, ED produced will meet in some common point F; and because the angles ABC, DCE are equal, BF is parallel to CD (*c*); also, because ACB equals DEC, AC is parallel to FE; therefore FACD is a parallelogram, whose opposite sides AC, DF, and AF, CD are equal (*d*).

And, in the triangle EBF, because AC is parallel to FE, and CD parallel to BF; therefore,

BA : AF, or its equal CD : : BC : CE, and
BC : CE : : FD, or its equal AC : DE (*e*).

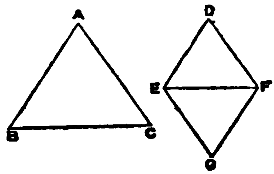
And taking both these alternately (*f*),

BA : BC : : CD : CE; also
BC : CA : : CE : ED; and again *ex æquo;*
BA : AC : : CD : DE, (*g*).

Wherefore, the sides about the angles, &c. Q. E. D.

Recite (*a*) p. 32, 1; (*b*) p. 17, 1; (*c*) p. 28, 1;
(*d*) p. 34, 1; (*e*) p. 2, 6; (*f*) p. 16, 5;
(*g*) p. 22, 5.

5 Th. If the sides of two triangles about each of their angles be proportionals, the triangles shall be equiangular, and shall have their equal angles opposite to homologous sides.




Given two triangles ABC, DEF; so that AB : BC : : DE : EF,
BC : CA : : EF : FD, and *ex æquali* AB : CA : : DE : FD;
then the triangles are equiangular, having their equal angles opposite to homologous sides

At the points E, F, in the straight line EF, make the angles FEG, EFG severally equal to the angles B, C (*a*); then the third angles G and A are equal (*b*)

Now, because the triangles ABC, EGF are equiangular, their sides about equal angles are proportionals (*c*); wherefore AB : BC : : GE : EF; but AB : BC : : DE : EF, by hyp., and so, DE : EF : : GE : EF, and therefore GE equals DE (*d*). For the same reason GF equals DF.

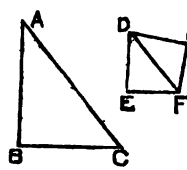
And since, in the triangles DEF, GEF, the sides EG, ED are equal, and EF is common, there are two sides in the one equal to two sides in the other; and the bases FG, FD are equal; therefore the angles DEF, GEF are equal; as are also the other angles, namely, DFE to GFE and D to G (*e*).

And because DEF equals GEF, it also equals B (*f*): for the same reason the angle C equals DFE, and the angle A equals the angle D. Therefore the triangles ABC and DEF are equiangular.

If therefore, the sides of two triangles, &c. Q. E. D.

Recite (*a*) p. 23, 1; (*b*) 32, 1; (*c*) p. 4, 6;
(*d*) p. 9, 5; (*e*) p. 8, 1; (*f*) ax. 1, 1.

6 Th. If two triangles have one angle of the one equal to one of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite homologous sides.




Given two triangles ABC, DEF, having the angles A and D equal, and BA to AC as ED to DF; then shall the other angles be equal; namely, B to E and C to F.

At the points D, F, in the side DF, make the angle FDG equal to FDE, or A, and DFG equal to C (*a*); the third angles G and B are equal (*b*).

Now, since the triangles ABC, DGF are made equiangular, BA : AC : : GD : DF (*c*); but BA : AC : : ED : DF by hyp.; therefore ED : DF : : GD : DF (*d*); and so, ED equals GD, (*e*), and DF is common to the two triangles EDF, GDF; therefore the sides ED, DF equal the sides GD, DF, and they contain equal angles: where-

fore ($f$) the bases EF, GF are equal; the angles E, G are equal; also DFE equals DFG. But G is made equal to B, and DFG to C; therefore B is equal to E, and C to DFE ($g$).

Wherefore, if two triangles have, &c. Q. E. D.

Recite ($a$) p. 23, 1; ($b$) p. 32, 1; ($c$) p. 4, 6;
($d$) p. 11, 5; ($e$) 9, 5; ($f$) p. 4, 1
($g$) ax. 1, 1.

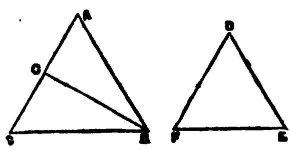
7 Th. If two triangles have two angles equal, and the sides about other two angles proportionals; then, whether the third two be oblique or right angles, the triangles shall be equiangular, and have those angles equal which are contained by the proportional sides.

Let two triangles ABC, DEF, have equal angles at A and D, and have AB to BC as DE to EF; their angles at B and E are equal, also those at C and F.

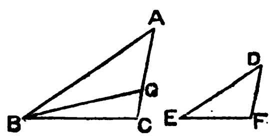
If then it be said, that the angles at B and E are unequal, make the angle ABG equal to E ($d$); and it follows, that AB : BG : : DE : EF ($b$); that AB has to BC and BG the same ratio; that BG equals BC ($c$), and that the angles BGC and BCG are equal ($d$).

Then since the angles at C and F may be right or oblique:

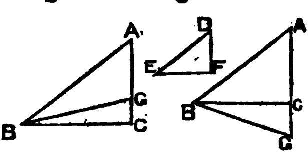
1. If C and F be acute, BGC is acute ($d$), and the adjacent angle BGA is obtuse ($e$); and so, F is obtuse, because the triangles ABG, DEF assume to be equiangular.



2. If C and F be obtuse, BGC is obtuse ($d$), and the adjacent angle BGA is acute ($e$); and so, F is acute, because the triangles ABG, DEF assume to be equiangular.




3. If C and F be right angles, BGC is a right angle ($d$) and so two angles of a triangle are not less than two right angles ($f$).

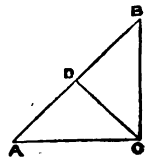



Therefore to deny the equality of the angles at B and E, makes the same angle F acute and obtuse, and two angles of a triangle equal to two right angles, which are both absurd.

Wherefore, if two triangles have two angles equal, &c.
Q. E. D.

Recite ($a$) p. 23, 1; ($b$) p. 4. 6; ($c$) p. 9, 5;
($d$) p. 5, 1; ($e$) p. 13, 1; ($f$) p. 17, 1.

**8 Th. In a right angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole and to each other.**




Given the triangle ABC, right angled at C: the perpendicular CD divides ABC into two triangles similar to the whole and to each other.

Because the angle BCA equals BDC (*a*), and that the angle B is common to the two triangles BCA, BDC, the third angles BAC, BCD, are equal to each other (*b*); therefore the triangles BCA and BDC are equiangular, and the sides about the equal angles are proportionals (*c*): therefore the triangles are similar (*d*).

In like manner, it may be shown, that the triangles ADC, BDC are equiangular and similar: and the triangles BDC, ADC, being each equiangular and similar to BCA, are equiangular and similar to each other.

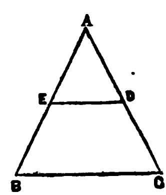
Therefore, in a right angled triangle, &c. Q. E. D.

Recite (*a*) ax. 10, 1; (*b*) p. 32, 1; (*c*) p. 4, 6; (*d*) def. 1, 6.

Cor. From the equiangular triangles, as above, come the following proportions namely, BD : BC : : BC : BA,
AD : AC : : AC : AB, and BD : DC : : CD : DA.

Therefore each side of the triangle ABC is a mean proportional between its adjacent segment of the base and the base complete; and the perpendicular is a mean proportional between the segments of the base

**9 P. From a given straight line (AB), to cut off any part required.**



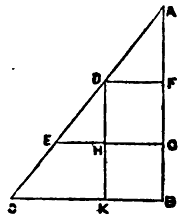

From the point A draw AC, at any angle with AB. In AC take a point D, and make AC the same multiple of AD that AB is of the part to be cut off from it: join BC, and draw DE parallel to it; AE is the part required.

Because ED is parallel to BC, a side of the triangle ABC (*a*), CD : DA : : BE : EA; and by composition, CA : AD : : BA : AE (*b*): but CA is a multiple of AD, and BA is the same multiple of AE (*c*): therefore, whatever part AD is of AC, the same part is AE of AB.

Wherefore, from the straight line AB the required part is cut off; which was to be done.

Recite (*a*) p. 2, 6; (*b*) p. 18, 5; (*c*) p. D, 5,

10 P. To divide a given straight line into parts, having the same ratios as the parts of a divided straight line given.




Given the straight line AC, divided in D, E; and AB to be divided into similar parts, or having the same ratios.

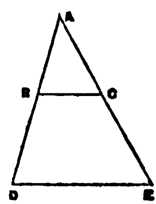
Place AB, AC, so as to contain any angle; join BC; and through the points D, E draw DF, EG parallel to BC (*a*); and through D draw DHK parallel to AB.

Now, in the parallelograms FH, HB, the side DH is equal to FG, and HK to GB (*b*). And because, in the triangle DKC, HE is parallel to KC, CE is to ED as KH is to HD, or as BG is to GF: and because, in the triangle AGE, FD is parallel to GE, ED is to DA as GF is to FA (*c*). And since it proves that CE : ED : : BG : GF, and ED : DA : : GF : FA;

Therefore, the given straight line AB is divided similarly to AC; which was to be done.

Recite (*a*) p. 31, 1; (*b*) p. 34, 1; (*c*) p. 2, 6.

11 P. To find a third proportional to two given straight lines, AB, AC.



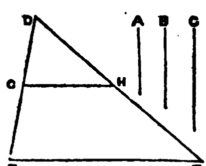

Place the given lines AB, AC so as to contain any angle; join BC; produce AB, so that BD shall equal AC; through D, draw DE parallel to BC (*a*), and produce AC to meet DE (*b*).

Because, in the triangle ADE, BC is parallel to DE, we have AB to BD, or to its equal AC, as AC to CE (*c*).

Therefore, CE is the third proportional to AB, AC; which was sought.

Recite (*a*) p. 31, 1; (*b*) pos. 2; (*c*) p. 2, 6.

12 P. To find a fourth proportional to three given straight lines A, B, C.




Place two straight lines DE, DF so as to contain any angle EDF; and upon these make DG=A, GE=B, and DH=C; join GH, and produce DH to meet EF drawn parallel to GE. (*a*).

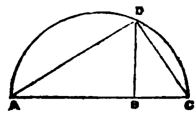
And because, in the triangle DEF, GH is parallel to EF, there is DG to GE as DH to HF (*b*): but DG, GE, DH were made equal to A, B, C:

Therefore, A : B : : C : HF; and so, HF is the fourth proportional sought.

Recite (*a*) p. 31, 1; (*b*) p. 2, 6.

13 P. To find a mean proportional between two given straight lines AB, BC.

Place AB, BC in a straight line AC; bisect it (*a*); and, upon the bisectional point, describe a semicircle to pass through A, C; through B draw BD at right angles to AC (*b*); join AD, CD.



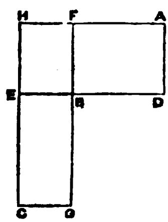

Because the angle ADC, in a semicircle, is a right angle (*c*); and that, in the right angled triangle ADC, a perpendicular BD is drawn from the right angle to the base; DB is a mean proportional between the segments of the base, which are the given straight lines AB, BC (*d*):

Wherefore, BD is the mean proportional sought.

Recite (*a*) p. 10, 1; (*b*) p. 11, 1; (*c*) p. 31, 3; (*d*) Cor. p. 8, 6.

14 Th. Equal parallelograms, which have equal angles, two and two, have their sides about equal angles reciprocally proportional: and parallelograms are equal, which have equal angles, two and two, and their sides about equal angles reciprocally proportional.

Given two equal parallelograms AB, BC, having equal angles at B. Place the parallelograms so that the angles at B shall be vertical to each other (*a*), and reciprocal sides, DB, BE and GB, BF, in straight lines: complete the parallelogram FE. Then DB : BE : : GB : BF; and if so, AB equals BC.




1. Because the parallelograms AB, BC are equal; and FE is another parallelogram; AB : FE : : BC : FE (*b*). But AB : FE : : DB : BE, and BC : FE : : GB : BF (*c*); therefore, DB : BE : : GB : BF (*d*); and these proportionals are reciprocal sides about equal angles.

2. Because DB : BE : : GB : BF; and that DB : BE : : AB : FE, and GB : BF : : BC : FE (*c*); therefore AB : FE : : BC : FE (*d*); and so, the parallelograms AB and BC having the same ratio to the parallelogram FE, are equal to each other (*e*).

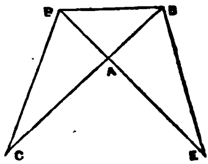
Therefore, equal parallelograms, &c. Q. E. D.

Recite (*a*) p. 15, 1; (*b*) p. 7, 5; (*c*) p. 1, 6; (*d*) p. 11, 5; (*e*) p. 9, 5.

NOTE.—The opposite sides and angles of a parallelogram are equal: p. 34 of b. 1.

15 Th. Equal triangles which have an angle of one equal to an angle of the other, have their sides about the equal angles reciprocally proportional: and triangles are

equal, which have an angle of one equal to an angle of the other, and the sides about the equal angles reciprocally proportional.




Given two equal triangles ABC, ADE, having equal angles at A. Place the triangles so that the angles at A may be vertical (*a*) to each other, and reciprocal sides CA, AD and EA, AB in straight lines: join BD. Then 1, CA : AD : : EA : AB; 2, ABC=ADE.

1. Because the triangles ABC, ADE are equal; and that ABD is another triangle, ABC : ABD : : ADE : ABD (*b*): but ABC : ABD : : CA : AD; and ADE : ABD : : EA : AB (*c*); therefore CA : AD : : EA : AB (*d*); and, taken in this order, these proportionals are reciprocal sides.

2. Because CA : AD : : EA : AB; and that CA : AD : : ABC : ABD, and EA : AB : : ADE : ABD (*c*); therefore ABC : ABD : : ADE : ABD (*b*); and so, the triangles ABC, ADE, having the same ratio to ABD, are equal to each other (*d*).

Therefore, equal triangles, &c. Q. E. D.

Recite (*a*) p. 14, 15, 1; def. 2, 6; (*b*) p. 7, 5;
(*c*) p. 1, 6; (*d*) p. 11, 5; (*e*) p. 9, 5.

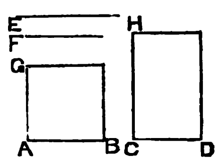
NOTE.—Reciprocal sides, def. 2, b. 6.

16 Th. If four straight lines be proportionals, the rectangles of the extremes and means are equal: and if the rectangles of the extremes and means be equal the four straight lines are proportionals.

1. Given AB to CD as E to F; sought AB×F=CD×E.
2. Given F×AB=E×CD: sought AB : CD : : E : F.

From the points A, C, and at right angles to AB, CD, draw AG=F, and CH=E (*a*).

Complete the rectangles BG, DH.



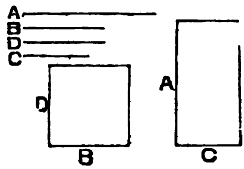

1. Because AB : CD : : E : F, and that E, F are equal to CH, AG; therefore AB : CD : : CH : AG (*b*). Therefore the sides of the parallelograms BG, DH, are reciprocally proportional; and so, BG is equal to DH (*c*). Now, BG is the rectangle of AB and AG, or F; and DH is the rectangle of CD and EH, or E: therefore AB×F=CD×E.

2. Because F times AB equals E times CD; and that E, F are equal to CH, AG: therefore AB×AG=CD×CH. But parallelograms that are equal and equiangular, have their sides about equal angles reciprocally proportional (*c*): therefore AB : CD : : CH : AG; that is, AB : CD : : E : F.

Therefore, if four straight lines, &c. Q. E. D.

Recite (*a*) p. 11, 1; (*b*) p. 7, 5; (*c*) p. 14, 6.

**17 Th.** If three straight lines be proportionals, the rectangle of the extremes is equal to the square of the mean: and if the rectangle of the extremes be equal to the square of the mean, the three straight lines are proportionals.




Given three straight lines A, B, C; such, that A is to B as B is to C: then A×C=B squared.

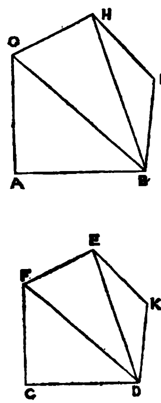
The square of B has two dimensions, each equal to B: let D=B. Then A : B : : D : C (*a*). But if four straight lines be proportionals, the rectangle of the means equals that of the extremes (*b*): therefore A×C=B×D. But the rectangle of B and D is the square of B; because B=D. Therefore the rectangle of A, C equals the square of B.

But let the rectangle of A, C equal the square of B; then A : B : : B : C. For, if B=D, as before, then A×C=B×D: and if the rectangle of the extremes be equal to that of the means, the four straight lines are proportionals; therefore A : B : : D : C (*b*). But B=D; therefore A is to B as B is to C.

Wherefore, if three straight lines, &c. Q. E. D.

Recite (*a*) p. 7, 5; (*b*) p. 16, 6.

**18 P.** Upon a given straight line (AB) to describe a rectilineal figure, similar and similarly situated to a given rectilineal figure (CDEF, or CDKEF).




Join DF; and make angles (*a*) at the points—
A, B; namely, A=C, and ABG=CDF,
B, G; namely, GBH=FDE, and BGH=DFE,
B, H; namely, HBL=EDK, and BHL=DEK.

Therefore, since the triangles ABG, CDF have two angles in the one equal to two in the other, they are equiangular (*b*); and their sides about the equal angles are proportional (*c*): the same is true of the triangles BGH, DFE; also of the triangles BHL, DEK.

Therefore AB : BG : : CD : DF (*c*),
and BG : BH : : DF : DE (*c*): Therefore *ex æquo* AB : BH : : CD : DE(*d*); which are sides of the trapezia, about equal angles.

It may also be proved in like manner, that the other sides of the trapezia, about their equal angles, are proportionals; also, that the sides of the irregular pentagons, about their equal angles, are proportionals.

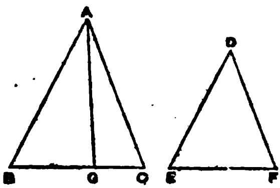
Therefore, the trapezium ABHG, and the irregular pentagon ABLHG, are severally similar

to the trapezium CDEF and pentagon CDKEF (e); and they are described upon the given straight line AB, as required.

And, in this manner, a figure of six or more sides, and similar to a given figure, may be described upon a given straight line.

Recite (a) p. 23, 1; (b) p. 32, 1; (c) p. 4, 6; (d) p. 22, 5; (e) def. 1, 6 and def. 35, 1.

19 Th. Similar triangles are to one another in the duplicate ratio of their homologous sides.




Given the similar triangles ABC, DEF; having the angles at B, E, equal, and AB to BC as DE to EF: then BC, EF are homologous; because they are the consequents of the ratios (a); and the ratio of ABC to DEF is the duplicate or square o the ratio of BC to EF (b).

Take BG, a third proportional to BC, EF (c); so that BC : EF : : EF : BG, and join GA. Then, since AB : BC : : DE : EF, alternately, AB : DE : : BC : EF (d). But BC : EF : : EF : BG; therefore AB : DE : : EF : BG (e): wherefore the triangles ABG, DEF are equal (f).

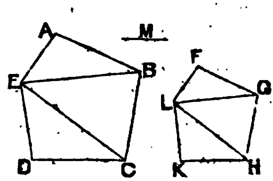
And since BC, EF, BG are three proportionals, the ratio of BC to BG is the duplicate of the ratio of BC to EF (b): but BC : BG : : ABC : ABG (g), or its equal DEF (h); wherefore the ratio of ABC to DEF is the duplicate of the ratio of BC to EF.

Therefore, similar triangles, &c. Q. E. D.

Recite (a) def. 10, 5; (b) def. 8, 5; (c) p. 11, 6; (d) p. 16, 5; (e) p. 11, 5; (f) p. 15, 6; (g) p. 1, 6; (h) p. 7, 5.

Cor. Hence, if three straight lines be proportionals, the first is to the third, as any triangle upon the first is to a similar triangle upon the second. The same is true of similar parallelograms, p. 41, 1.

20 Th. Similar polygons may be divided into the same number of similar triangles, having to each other the ratio of the polygons; which is the duplicate ratio of their homologous sides.

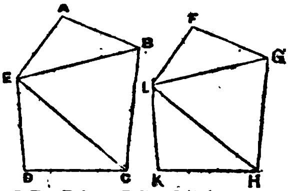



Let two polygons ABCDE, FGHKL, be similar, and have the sides AB, FG homologous (a).

Draw EB, EC and LG, LH; which divide the polygons into three triangles each.

Now these triangles are to be proved similar.

1. In the similar polygons (b), AB : AE : : FG : FL; and the an-




gles are equal at A, F; so the triangles ABE, FGL being equal (*c*), have the angles ABE, FGL equal. But these equals are parts of the equal polygonal angles at B, G; therefore the other parts EBC, LGH are equal (*d*).

Also, in these equiangular triangles EB : BA : : LG : GF (*e*);
And in the similar polygons AB : BC : : FG : GH (*b*);
Therefore, from equal distance EB : BC : : LG : GH (*f*).

Hence the triangles EBC, LGH, have their sides about equal angles proportional, and are therefore similar: for like reason, the triangles ECD, LHK are similar. Therefore, the similar polygons are cut into an equal number of similar triangles.

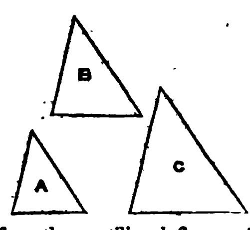
Again, the similar triangles have to each other the polygonal ratio; namely, the duplicate ratio of AB to FG. For the ratio—
of ABE to FGL is the duplicate of BE to GL (*g*)
of BCE to GHL is the duplicate of BE to GL; also
of BCE to GHL is the duplicate of CE to HL, and
of CDE to HKL is the duplicate of CE to HL.

Therefore, the ratios of ABE to FGL, BCE to GHL, and CDE to HKL are all equal; and, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents (*h*). Thus the ratio of ABE to FGL equals that of ABCDE to FGHKL; which is the duplicate ratio of the homologous sides AB, FG; and so of the other triangles.

Wherefore, similar polygons may be divided, &c. Q. E. D.

Recite (*a*) def. 11, 5; (*b*) def. 1, 6; (*c*) p. 4, 1;
(*d*) ax. 3, 1; (*e*) p. 5, 6; (*f*) p. 22, 5;
(*g*) p. 19, 6; (*h*) p. 12, 5.

21 Th. Rectilineal figures which are similar to the same rectilineal figures are also similar to one another




Let each of the figures A, B, be similar to C; A is similar to B.

For since A is similar to C, they are equiangular, and have their sides about equal angles, proportionals (*a*).

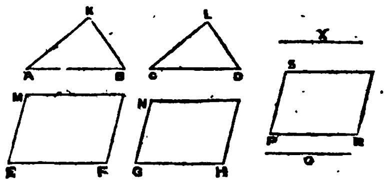
And since B is similar to C, they are equiangular, and have their sides about equal angles proportionals (*a*).

Therefore the figures A, B, are, each of them, equiangular to C, and have the sides about equal angles of each of them, and of C proportionals. Wherefore, the rectilineal figures A and B are equiangular (*b*), and have their sides about equal angles proportionals (*c*).

Therefore, A is similar to B Q. E. D.

Recite (*a*) def. 1, 6; (*b*) ax. 1, 1; (*c*) p. 11, 5.

22 Th. If four straight lines be proportionals, the similar rectilineal figures described upon them shall be proportionals: and if the similar rectilineal figures described upon four straight lines be proportionals, those lines shall also be proportionals.




1. Let AB, CD, EF, GH, be four straight lines, of such kind that AB : CD : : EF : GH; and upon AB, CD, describe similar figures, ABK, CDL; also, upon EF, GH, describe similar figures FM, HN.

Then ABK : CDL : : FM : HN.
To AB, CD let X be a third proportional;
and to EF, GH let O be a third proportional (*a*).

And since AB : CD : : EF : GH by hyp.
and CD : X : : GH : O, (*b*);
*ex æquali* AB : X : : EF : O, (*c*).

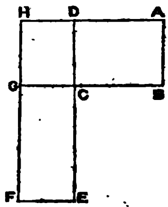
But AB : X : : ABK : CDL (*d*);
and EF : O : : FM : HN (*d*);
therefore ABK : CDL : : FM : HN (*d*), as stated.

2. If ABK be to CDL as FM is to HN; then AB : CD : : EF : GH.

Let AB be to CD as EF is to PR (*e*); and upon PR describe the rectilineal figure RS, similar to HN, or FM (*f*). Then since AB : CD : : EF : PR; and upon AB, CD similar figures ABK, CDL, are described; and upon EF, PR similar figures FM, RS, are described; therefore ABK : CDL : : FM : RS. But, by hyp. ABK : CDL : : FM : HN; therefore FM has to RS and HN the same ratio; and so, RS equals HN (*g*); therefore, being equal and similar, their sides about equal angles are equal: hence PR equals GH; and for this cause, and that AB : CD : : EF : PR, it is clear that AB : CD : : EF : GH. Therefore, if four, &c. Q. E. D

Recite (*a*) p. 11, 6; (*b*) p. 11, 5; (*c*) p. 22, 5;
(*d*) cor. p. 19, 6; (*e*) p. 12, 6; (*f*) p. 18, 6;
(*g*) p. 9, 5.

23 Th. Equiangular parallelograms have to each other the ratio which is composed of the ratios of their sides:— the sides of the antecedent being the antecedents, and the sides of the consequent being the consequents.




Given AC, CF, two parallelograms, having equal angles at C: the ratio of AC to CF is composed of the ratios of BC to CD and of EC to CG.

Place the angles at C vertical to each other (*a*); then the sides adjacent to them shall be in straight lines (*b*): complete the parallelogram DG.

Now, since parallelograms of the same altitude are to each other as their bases (*c*),

AC : DG : : BC : CG,

and DG : CF : : DC : CE.

And, since in compound ratios, the product of the antecedents is the antecedent, and the product of the consequents is the consequent (*d*);

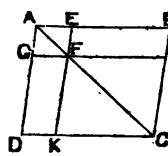
Therefore AC$\times$DG : DG$\times$CF : : BC$\times$DC : CG$\times$CE.

But magnitudes have the same ratio as their equimultiples (*e*); therefore AC : CF : : BC$\times$DC : CG$\times$CE.

Wherefore, equiangular parallelograms, &c. Q. E. D.

Recite (*a*) p. 15, 1; (*b*) p. 14, 1; (*c*) p. 1, 6;
(*d*) def. 10, and p. 22, 5; (*e*) p. 15, 5

24 Th. The parallelograms about the diameter of any parallelogram, are similar to the whole and to one another.




Given the whole parallelogram BD, of which the diameter is AC; and EG, HK, parallelograms about the parts of AC: the three are similar to each other.

1. BD and EG are equiangular: for since DC, GF are parallels, the angles ADC, AGF are equal (*a*); and since BC, EF are parallels, the angles ABC, AEF are equal (*a*); also the common angle DAB equals each of the opposite angles BCD, EFG (*b*).

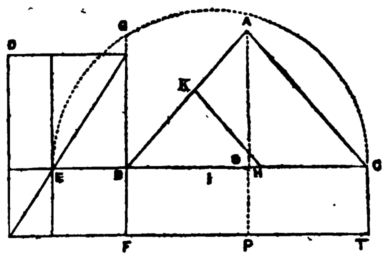
2. The sides of BD, EG, about equal angles, are proportionals; for, in the triangles BAC, EAF, the angles ABC, AEF are equal, and BAC is common: the triangles are therefore equiangular; and so, AB : BC : : AE : EF (*c*). And since the opposite sides of parallelograms are equal (*b*), AB : AD : : AE : AG, and DC : CB : : GF : FE, also CD : DA : : FG : GA (*d*).

Therefore the parallelograms BD, EG are equiangular; and the sides about their equal angles are proportional: they are therefore similar to each other, as stated (*e*). For the same reason, BD is similar to HK; wherefore, also, EG is similar to HK (*f*).

Wherefore, the parallelograms, &c. Q. E. D

Recite (*a*) p. 29, 1; (*b*) p. 34, 1; (*c*) p. 4, 6;
(*d*) p. 7, 5; (*e*) def. 1, 6; (*f*) p. 21, 6.

**25 P. To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure.**




Given the rectilineal figures ABC and DE: it is required to describe a rectilineal figure similar to ABC, and equal to DE.

Draw AO perpendicular to BC (*a*); produce AO to P, so that OP be equal to half AO; through P draw FT parallel to BC (*b*); complete the parallelogram BT; to the straight line BF apply the parallelogram EF (*c*), making EF equal to DE (*d*); bisect the straight line CE in I (*e*); on the centre I describe the semicircle CGE: through B draw BG at right angles to CE (*f*); make BH equal to BG, and upon it describe the triangle BHK equiangular to ABC (*g*). BHK is equal to DE.

DE is equal to EF; for they are complements about the diameter of a parallelogram (*d*): but BT : EF : : BC : EB (*h*); and because CGE is a semicircle, BG is a mean proportional between BC and EB (*i*); therefore BC is to EB as the triangle ABC is to a similar triangle described upon BG (*k*), or upon its equal BH; that is, BHK. And since BC : EB : : BT : EF, or ABC : BHK; therefore ABC : BHK : : BT : EF (*l*). But ABC equals BT; for they have the same base, and the triangle has two altitudes of the parallelogram (*m*); therefore BHK is equal to EF (*n*), or DE, as stated.

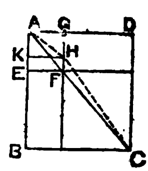
Wherefore, because BHK is equiangular to ABC, it is also similar to it (*o*); and it also proves equal to DE.

Therefore, a rectilineal figure has been described, &c.

Q. E. F.

Recite (*a*) p. 12, 1; (*b*) p. 31, 1; (*c*) p. 45, 1;
(*d*) p. 43, 1; (*e*) p. 10, 1; (*f*) p. 11, 1;
(*g*) p. 18, 6; (*h*) p. 1, 6; (*i*) p. 13, 6;
(*k*) cor. p. 19, 6; (*l*) p. 11, 5; (*m*) p. 41, 1;
(*n*) p. 14, 5; (*o*) p. 4, 6.

**26 Th. If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.**



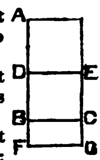

Given the parallelograms BD, EG, similar, and having the angle A common, they are about the same diameter.

For, if BD have for its diameter AHC, and EG have AF, let GF meet AHC in H; draw KH parallel to AD, or BC: then the parallelograms BD, KG, being about the same diameter, are similar to one another (*a*): wherefore AB : AD : : AK : AG

(*b*): but because BD, EG are similar, AB : AD : : AE : AG (*c*); and so, AE : AG : : AK : AG, and AE equals AK (*d*); but it is evidently greater. Therefore AHC is not the diameter of BD; nor is any other line, that is not also the diameter of EG: therefore BD and EG are about the same diameter. Q. E. D.

Recite (*a*) p. 24, 6; (*b*) p. 1, 6;
(*c*) p. 11, 5; (*d*) p. 9, 5.

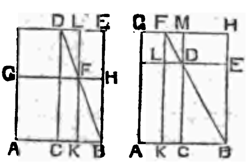
## *Definitions.*




1. A parallelogram is exactly applied to a straight line, when that line makes one of its sides; as AC to AB.
2. A parallelogram is defectively applied to a straight line, when that line makes more than one of its sides; as AE to AB, where the defect is DC.
3. A parallelogram is excessively applied to a straight line, when that line makes less than one of its sides; as AG to AB, where the excess is BG.

27 Th. The greatest of all defective parallelograms applied to the same straight line, is that described upon its half; provided each of the defects be similar to the parallelogram described upon half the line.

Given the straight line AB bisected in C, and the figures completed, as in the margin; the defective parallelogram AD, whose defect is similar to CE (for it is CE), described upon half the line, is greater than AF, whose defect KH, is similar to CE, because they are about the same diameter BD (*a*).



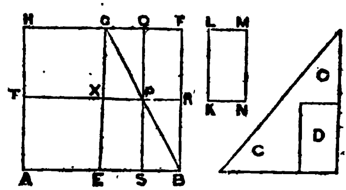

1. Let AK, the base of AF, exceed AC; because CF equals FE (*b*), add KH to each; therefore the whole CH is equal to the whole KE. But CH is equal to CG (*c*); therefore CG equals KE: to each of these add CF; therefore the parallelogram AF equals the gnomon CHL, and is therefore less than CE, or AD, as stated.

2. Let AK, the base of AF, be less than AC: now DH equals DG (*c*); for HM equals MG, because BC equals AC: wherefore DH exceeds LG: but DH equals DK (*b*); therefore DK exceeds LG: to each add AL; therefore the whole AD exceeds the whole AF, as stated.

Wherefore, the greatest of all defective, &c. Q. E. D.

Recite (*a*) p. 26, 6; (*b*) p. 43, 1; (*c*) p. 36, 1.

28 P. To a given straight line to apply a defective parallelogram equal to a given triangle, whose defect is similar to a given parallelogram; but the given triangle must not exceed the parallelogram applied to half the given line, whose defect is similar to that of the parallelogram to be applied.




Given the straight line AB; also the triangle CC, equal to the defective parallelogram to be applied to AB, but not greater than the parallelogram applied to half AB, and D the figure to which both defects are to be similar.

Bisect AB in E (*a*), and upon EB describe the parallelogram EF similar to D (*b*), and complete the parallelogram AG; which, if equal to CC, is the thing required: for AG is applied to AB, defective by a parallelogram similar to D.

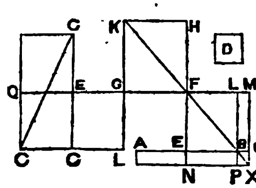
But if AG be not equal to CC, it is greater by hypothesis; and so, EF exceeds CC. Make (*c*) the parallelogram KM equal to the excess of EF above CC, and similar to D, or EF (*d*); and let EG be homologous to KL, and GF to LM. Therefore EF, being equal to KM and CC both, is greater than KM; and its sides EG, GF exceed the sides KL, LM. Make GX equal to LK, and GO to LM, and complete the parallelogram XO; which is equal and similar to KM, and similar to EF; and EF, XO are about the same diameter (*e*). Draw the diameter BG and complete the scheme.

Now, because EF equals KM and CC, or XO and C, the gnomon ERO equals the part CC; but EP equals RO (*f*): to each add RS; the sums ER, OB are equal: but ER equals ET (*g*); therefore ET equals OB: to each add EP; then the sum ST equals the gnomon ERO, which proved equal to CC.

Therefore ST equals CC; and it is applied to AB, defective by a parallelogram RS, similar to D; because RS and EF are about the same diameter (*h*): which was to be done.

Recite (*a*) p. 10, 1; (*b*) p. 18, 6; (*c*) p. 25, 6;
(*d*) p. 21, 6; (*e*) p. 26, 6; (*f*) p. 43, 1;
(*g*) p. 36, 1; (*h*) p. 24, 6.

29 P. To a given straight line to apply an excessive parallelogram equal to a given triangle: the excess to be similar to a given parallelogram.




Given AB, a straight line; CC, a triangle equal to the excessive parallelogram to be applied to AB; and D, a parallelogram to which the excess is to be similar.

Bisect AB in E (*a*); upon EB describe the parallelogram EL, similar to D (*b*); make the parallelogram GH equal to EL and CC both (*c*), and similar to D, or EL (*d*). Let KH be the side homologous to FL, and KG to FE: and because GH exceeds EL, its sides KH, KG exceed the sides FL, FE. Make FM equal to KH, and FN equal to KG; and complete the scheme MNA.

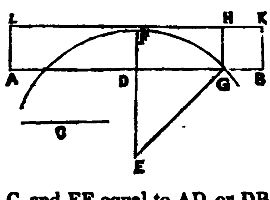
MN is equal and similar to GH; and therefore similar to EL; and so, MN, EL are about the same diameter FX (*e*). Therefore MN being equal to GH, is equal to EL and CC both: and its gnomonic part NOL, is therefore equal to the triangle CC. But NA equals NB (*f*), or its equal BM (*g*); therefore the gnomon NOL, or its equal CC, is equal to the parallelogram AX; which is applied to AB; having its excess OP similar to EL, since they are about the same diameter (*h*); and EL is similar to D; wherefore OP is also similar to D, as required.

Therefore to a given straight line is applied, &c., which was to be done.

| | | |
|---|---|---|
| Recite (*a*) p. 10, 1; | (*b*) p. 18, 6; | (*c*) p. 25, 6; |
| (*d*) p. 21, 6; | (*e*) p. 26, 6; | (*f*) p. 36, 1; |
| (*g*) p. 43, 1; | (*h*) p. 24, 6. | |

*The four following Problems are taken from Simson's Notes:*

Note 1. To apply a defective rectangle equal to a given square to a given straight line, the defect to be a square: but the given square must not exceed that upon half the given line.




Given the straight line AB; also the square of C, equal to the rectangle to be applied, but not greater than the square of half AB.

Bisect AB in D; then if C equal AD, their squares are equal, and the defect is the square of DB.

But if C be less, for it may not be greater, than AD, draw DE at right angles to AB (*a*); make DE equal to C, and EF equal to AD, or DB; from E as centre, with the distance EF, describe an arc, meeting AB in G; upon GB describe the square BH (*b*); complete the rectangle AH, and join EG.

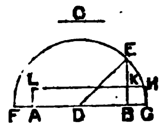
Now the rectangle AG×GB, with the square of DG, is equal to the square of AD, or its equal EG (*c*): but the square of EG equals the squares of ED, DG (*d*); therefore, the rectangle AG×GB, with the square of DG, equals the squares of ED and DG. Take from

both the square of DG: the rectangle AG×GB remains equal to the square of ED, or C.

Wherefore, to a given straight line AB, a rectangle is applied equal to a given square and defective by a square; which was to be done.

Recite (*a*) p. 11, 1; (*b*) p. 46, 1; (*c*) p. 5, 2; (*d*) p. 47, 1.

Note 2. To apply an excessive rectangle, equal to a given square, to a given straight line: the excess to be a square.



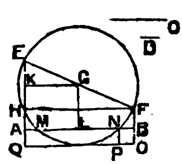
Given AB, a straight line, and the square of C, equal to the rectangle to be applied.

Bisect AB in D; draw BE at right angles to it, so that BE equal C; join DE, and with this as radius, describe a circle to meet AB produced in G; upon BG describe the square BH, and complete the rectangle BL.

Since AB is bisected in D, and produced to G, the rectangle AG×GB, with the square of DB, is equal to the square of DG (*a*), or DE its equal, or the squares of DB and BE (*b*). Take from both sides the square of DB; therefore the rectangle AG×GB equals the square of BE, or its equal C. But the rectangle AG×GB is the rectangle AH; which is applied to AB, exceeding by the square BH, as required to be done.

Recite (*a*) p. 6, 2; (*b*) p. 47, 1.

Note 3. To apply a defective rectangle to a given straight line, not greater than the square of its half, but equal to a given rectangle: the defect to be a square.



Given AB, a straight line; and C×D, a rectangle, equal to the defective one to be applied to AB, but not to exceed the square of half AB.

On one side of AB, draw AE, BF at right angles to it; so that AE=C, and BF=D: join EF, and bisect it in G: with GE radius, describe the circle EHF: join HF; make GK parallel to it, and GL parallel to AE.

Since the angle EHF, in a semicircle, is equal to the right angle EAB, HF is parallel to AB, and AH to BF: wherefore AH=BF, and EA×AH=EA×BF; that is, C×D.

And since EG=GF, and AE, LG, BF are parallels; therefore AL=LB, and EK=KH (*a*): and since C×D may not exceed $AL^2$, or $GK^2$, to each add $KE^2$, therefore C×D+$KE^2$; that is, EA×AH+$KH^2$, or $AK^2$ (*b*), is not greater than $GK^2$+$KE^2$, that is, $EG^2$ (*c*): consequently, AK, or GL, is not greater than EG.

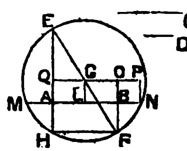
Now, if GE be equal to GL, the circle will touch AB in L (*d*); and $AL^2$ will equal EA×AH, or C×D, which is the thing required.

But if GE, GL be unequal, GE is greater; and the circle will cut

AB in points M, N. Upon NB describe the square NO, and complete the rectangle NQ. Now since AL=BL, and ML=NL (*a*); therefore AM=NB; and AN×NB=AN×AM; that is, AE×AH (*e*), or C×D. But AN×NB is the rectangle AP; which is applied to AB, defective by the square NO, as was required to be done.

Recite (*a*) p. 3, 3; (*b*) p. 6, 2; (*c*) p. 47, 1; (*d*) p. 36, 3; (*e*) cor. p. 36, 3.

Note 4. To apply an excessive rectangle to a given straight line, equal to a given rectangle: the excess to be a square.




Given AB, a straight line; and C×D a rectangle equal to the excessive one to be applied to AB.

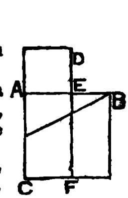
On the contrary sides of AB, at right angles to it, draw AE=C, and BF=D: join EF, and bisect it in G: with GE as radius, describe a circle EHF; join HF; draw GL parallel to AE; produce AB to meet the circle in points M, N; upon BN describe the square NO, and complete the rectangle BQ.

The angle EHF, in a semicircle (*a*), equals the right angle EAB; therefore HF is parallel to AB (*b*), and AH=BF (*c*); therefore EA×AH=EA×BF, or C×D.

And since ML=LN (*d*), and AL=LB; therefore MA=BN, and AN×NB=AM×AN, or EA×AH (*e*); that is, C×D. But AN×NB is the rectangle AP, which is equal to the given rectangle C×D, and applied to AB, exceeding by the square NO, as required.

Recite (*a*) p. 31, 3; (*b*) p. 28, 1; (*c*) p. 34, 1; (*d*) p. 3, 3; (*e*) p. 35, 3.

30 P. To cut a given straight line in extreme and mean ratio.




Given AB, a straight line, to be cut into two such parts, AE, EB, as that AB is to AE as AE is to EB.

Upon AB describe the square BC (*a*); and to AC, a side of the square, apply CD, an excessive rectangle, equal to BC: the excess being the square AD. See Note 4, above.

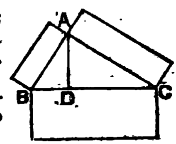
Then since BC=CD, and the part AF is common to both; removing this, the remainders AD and BF are equal: these figures are also equiangular; hence their sides are reciprocally proportional (*b*). Wherefore EF : ED : : AE : EB. But EF=AC, or AB (*c*); and ED=AE. Therefore AB : AE : : AE : EB,

as stated. But AB is greater than AE, therefore AE is greater than EB (*d*). Hence the straight line AB is divided in extreme and mean ratio, as required (*e*).

Recite (*a*) p. 46, 1 : (*b*) p. 14, 6 ; (*c*) p. 34, 1 ;
(*d*) p. 14, 5 ; (*e*) def. 3, 6.

Otherwise. Divide AB in the point E; so that $AE_2$, that is, AE×AE may equal AB×EB (p. 11, 2): then it will be that AB : AE : : AE : EB, &c., as above (p. 17, 6).

31 Th. In right angled triangles, the rectilineal figure described upon the side subtending the right angle, is equal to the figures similar to it, described upon the sides containing the same angle.




Given the triangle ABC, right-angled at A; the rectilineal figure described upon BC, is equal to the figures similar to it, described upon AB, AC.

Draw AD perpendicular to BC: then the triangles ADB, CDA, are similar to CAB and to each other (*a*).

1. Since CAB, ADB are similar, CB : AB : : AB : DB (*b*): the straight lines CB, AB, DB are therefore three proportionals; and CB : DB : : fig. upon CB : similar fig. upon AB (*c*).

2. Since CAB, CDA are similar, CB : CA : : CA : CD (*b*): the straight lines CB, CA, CD are therefore three proportionals; and CB : CD : : fig. upon CB : similar fig. upon CA (*c*).

Taking these two proportions inversely (*d*):

DB : CB : : fig. upon AB : fig. upon CB ; and
CD : CB : : fig. upon CA : fig. upon CB.

Adding 1st and 5th, 2d and 6th (*e*):

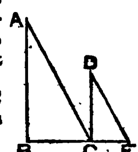
CD+DB : CB : : figs. upon CA+AB : fig. upon CB.

But CD+DB=CB; therefore figs. upon CA+AB=fig. upon CB (*f*).

Therefore, in right-angled triangles, &c. Q. E. D.

Recite (*a*) p. 8, 6 ; (*b*) p. 4, 6 : (*c*) cor. p. 19, 6 ;
(*d*) p. B, 5 ; (*e*) p. 24, 5 ; (*f*) p. A, 5.

32 Th. If two triangles, having two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to each other; their third sides shall be in one straight line.




Given two triangles ABC, DCE, joined at their point C, with BA parallel to CD, and AC to DE; also BA : AC : : CD : DE—then BC, CE are in one straight line.

Because AC cuts the parallels BA, CD; and DC cuts the parallels AC, DE; the three alternate angles A, ACD, D are equal to each other (*a*).

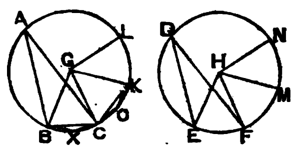
And because the angles A, D are equal, and the

sides about them proportionals, the triangles ABC, DCE are equiangular (*b*); therefore B=DCE. and ACB=E: moreover ACD=A. Therefore the angles ACB, ACD, DCE together, are equal to all the angles of the triangle ABC; that is, to two right angles (*c*); or half the compass of the angular point C: therefore BC, CE, form no angle with each other; but are in one straight line (*d*).

Wherefore, if two triangles, &c. Q. E. D.

Recite (*a*) p. 29, 1; (*b*) p. 6, 6; (*c*) p. 32, 1; (*d*) cor. def. 11, 1; also p. 14, 1.

33 Th. In equal circles, angles, either at the centres or circumferences, have the same ratio as the arcs have on which they stand: so also have the sectors.




Given two equal circles, ABC, DEF; in which the angles BGC, EHF are central, and BAC, EDF are at the circumference: the angles BGC, EHF, &c., as also the sectors, have the same ratio as the arcs BC, EF.

Take arcs CK, KL, each equal to BC; also arcs FM, MN, each equal to EF: join G to K, L, and H to M, N.

Then since the arcs BC, CK, KL are equal, their angles are equal (*a*); and the arc BL, and angle BGL, are equimultiples of the arc BC, and angle BGC.

And since the arcs EF, FM, MN are equal, their angles are equal (*a*); and the arc EN, and angle EHN, are equimultiples of the arc EF, and angle EHF.

If then the arc BL be equal to, greater, or less than the arc EN, the angle BGL is equal to, greater, or less than the angle EHN. BL and BGL being therefore equimultiples of BC the first, and BGC the third; and EN and EHN being equimultiples of EF the second, and EHF the fourth: it follows (*b*), that BC : EF : : BGC : EHF. Therefore the central angles have the ratio of their arcs, as stated.

But, because the central angles are second multiples of those at the circumference (*c*), BGC : EHF : : BAC : EDF (*d*); and so, BC : EF : : BAC : EDF (*e*); that is, the angles at the circumference have also the ratio of their arcs.

The sectors also have the ratio of their arcs. Join C to B, K. Then since the radii GB, GC, GK are equal, and contain equal angles, as above, the bases are equal; namely, the chords BC, CK; finally, the triangles GBC, GCK are equal (*f*). Moreover, the arcs were made equal, and the chords prove equal; therefore, the segments are similar and equal (*g*), the greater to the greater and the less to the less (*h*).

Now a sector, in this case, is composed of a triangle and one of the less segments; and these prove equal, each to each: therefore the sectors BGC, CGK are equal; and KGL may be proved equal to either, in the same way. Moreover, the sectors EHF, FHM, MHN may be proved equal to each other. Therefore, the sectors BGL,

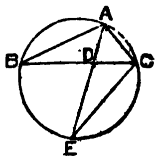
EHN are the same multiples of the sectors BGC, EHF, that the arcs BL, EN are of the arcs BC, EF.

If then the arc BL be equal to, greater, or less than the arc EN: the sector BGL is equal to, greater, or less than the sector EHN. BL and BGL being therefore equimultiples of BC the first, and BGC the third; and EN and EHN, being equimultiples of EF the second, and EHF the fourth; it follows, that BC : EF : : BGC : EHF (*b*): therefore the sectors have the same ratio as their arcs.

Wherefore, in equal circles, &c. Q. E. D.

Recite (*a*) p. 27, 3; (*b*) def. 5, 5; (*c*) p. 20, 3;
(*d*) p. 15, 5; (*e*) p. 11, 5; (*f*) p. 4, 1;
(*g*) p. 24, 3; (*h*) p. 28, 3.

B Th. If a straight line bisect an angle of a triangle, and likewise cut the base; the rectangle contained by the sides is equal to the rectangle contained by the segments of the base, together with the square of the bisecting line.




Given ABC a triangle, and AD a straight line bisecting the angle BAC: then $BA\times AC = BD\times DC + AD^2$.

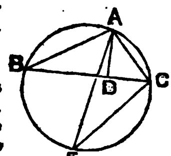
Describe the circle ACB about the triangle (*a*); produce AD to the circumference in E; join EC.

Then because the angles BAD, CAE were made equal; and ABD equals AEC, in the same segment (*b*); the triangles ABD, AEC are equiangular: therefore BA : AD : : EA : AC (*c*); hence $BA\times AC = EA\times AD$ (*d*), that is, $ED\times DA + AD^2$ (*e*). But $ED\times DA = BD\times DC$ (*f*). Therefore $BA\times AC = BD\times DC + AD^2$, as stated.

Wherefore, if a straight line bisect, &c. Q. E. D.

Recite (*a*) p. 5, 4; (*b*) p. 21, 3; (*c*) p. 4, 6;
(*d*) p. 16, 6; (*e*) p. 3, 2; (*f*) p. 35, 3.

C Th. If from any angle of a triangle a perpendicular be drawn to the base; the rectangle contained by the sides of the triangle equals that contained by the perpendicular and the diameter of the circle described about the triangle.




Given ABC a triangle, AD a perpendicular from the vertex A to the base, also AE the diameter of the described circle: then $BA\times AC = AD\times AE$. Join CE.

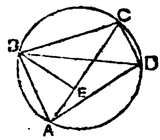
Because the angle ACE, in a semicircle, equals the right angle ADB (*a*); and the angles ABC, AEC, in the same segment, are equal (*b*); the triangles ABD, AEC are equiangular: therefore

BA : AD : : EA : AC (*c*); and it follows that $BA\times AC = EA\times AD$, (*d*), as stated.

Wherefore, if from any angle, &c. Q. E. D.

Recite (*a*) p. 31, 3; (*b*) p. 21, 3;
(*c*) p. 4, 6; (*d*) p. 16, 6.

D Th. The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.




Given ABCD any quadrilateral inscribed in a circle; also AC, BD, diagonals joining its opposite angles: AC×BD=(BC×AD)+(AB×CD).

Make the angle ABE equal to CBD; to each of which add DBE: the sums ABD, CBE are equal: and as angles in the same segment BDA equal BCE (*a*). Therefore the triangles ABD, CBE are equiangular; and BC : EC : : BD : DA (*b*); from which it comes that BC×DA=BD×EC (*c*).

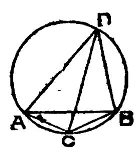
And because the angles ABE, DBC are made equal; and BAE equals BDC of the same segment (*a*); therefore the triangles ABE, DBC are equiangular; and BA : AE : : BD : DC (*b*); from which it comes that BA×DC=AE×BD (*c*).

Therefore, the two rectangles BD×EC and BD×AE are equal to the two BC×DA and BA×DC. But AE and EC are the parts of the diagonal AC; therefore BD×AC=(BD×AE)+(BD×EC) (*d*)= (BC×AD)+AB×CD).

Therefore, the rectangle, &c. Q. E. D.

Recite (*a*) p. 21, 3; (*b*) p. 4, 6; (*c*) p. 16, 6; *d*) p. 1, 2.

E Th. In a circle, the chord of an arc is to the chord of its half, as the sum of two straight lines drawn from the ends of the chord to any point in the opposite arc, is to the line which joins that point to the bisectional point of the arc.




Given the arc ACB bisected in C; also the chords AB, BC, and the straight lines AD, DB, DC: it is to be proved that AB is to BC or AC as AD+DB is to DC.

Since AB, DC are diagonals of a quadrilateral inscribed in a circle, the rectangle AB×DC equals the two rectangles AD×BC and DB×AC (*a*).

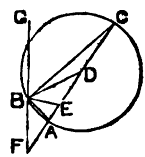
But since AC=BC (*b*), these two rectangles are equal to AD×AC and DB×AC, or (AD+DB) ×AC (*c*). Therefore, AB×DC=(AD+DB)×AC: and since the sides of equal rectangles are reciprocally proportional (*d*), it follows that AB is to AC, or BC as AD+DB is to DC, as stated.

Wherefore, in a circle, the chord of an arc, &c. Q. E. D.

Recite (*a*) p. D, 6; (*b*) p. 29, 3; (*c*) p. 1, 2; (*d*) p. 14, 6.

F Th. If three straight lines be proportionals; namely, 1, a line drawn from *a point* without a circle to the cen-

tre; 2, the radius; 3, a segment measured from the centre to a *point* in the first: then, any two lines drawn from *the two points* to the circumference shall have the same ratio as those two segments of the first line which meet the circumference, beginning at the said points.




Let F be the outer point, D the centre, E the sectional point, and A, B, C, points in the circumference: Draw FB and produce it to G; join BA, BC, BD, BE.

Now, since FD : DB : : DB : DE, by hyp., and that these proportionals are about the same angle D, the triangles FBD, BED are equiangular (*a*).

Therefore FB : BD : : BE : ED (*b*), and
alternately FB : BE : : BD, or AD : ED (*c*).
But since FD : AD : : AD : ED, by hyp.
by division FA : AD : : AE ; ED, (*d*);
and alternately FA : AE : : AD, or BD : ED (*c*):
Therefore FA : AE : : FB : BE (*e*).

But FA, AE are the segments of the first line, which meet the circumference; and FB, BE are any two lines drawn from the points to the circumference.

Wherefore, if three straight lines, &c. Q. E. D.

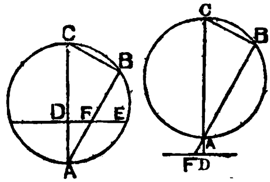
Recite (*a*) p. 6, 6; (*b*) p. 4, 6; (*c*) p. 16, 5;
(*d*) p. 17, 5; (*e*) p. 11, 5.

Cor. The chord AB bisects the angle FBE: for, in the triangle FBE, the segments of the base FA, AE have the same ratio as the sides FB, BE. Recite p. 3, 6.

And for the same reason the chord BC bisects the exterior angle EBG, (p. A, 6).

For since by hyp. FD : DC : : DC : DE
by composition FC : CD : : CE : DE (p. 18, 5);
and by division FA : AD or CD : : AE : DE (p. 17, 5:
Therefore, *ex æquo* FA : AE : : FC : CE (p. 22, 5).
But it proves that FA : AE : : FB : BE;
Therefore FB : BE : : FC : CE (p. 11, 5).

G Th. If a chord, drawn through the extreme point of the diameter of a circle, meet a straight line in the circle or out of it, cutting the diameter or its exterior production, at right angles; the rectangle of the diameter and its segment, between the extreme point and perpendicular, is equal to the rectangle of the chord and its segment, between the same point and the cutting line.



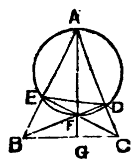

Given the chord AB and its segment AF; also the diameter AC, and its segment AD: then AB×AF=AC×AD. Join BC.

Because ABC is an angle in a semicircle (*a*); it is equal to the right angle ADF; and the angles DAF, BAC are either identical or vertical, and therefore equal (*b*): therefore the triangles ABC, ADF are equiangular (*c*); and BA : AC : : DA : AF. But the rectangle contained by the means is equal to that contained by the extremes (*d*); therefore DA×AC=BA×AF.

Wherefore, if a chord, &c. Q. E. D.

Recite (*a*) p. 31, 3; (*b*) p. 15, 1; (*c*) p. 4, 6; (*d*) p. 15, 6.

H Th. The perpendiculars drawn from the three angles of a triangle to the opposite sides, intersect each other in the same point.




Given the triangle ABC; perpendiculars BD, CE intersecting each other in F, and AF joined; which last produced to G will fall perpendicular to BC. Join ED.

Since AF subtends a right angle on either side of it, it is the diameter of a circle passing through the points A, E, F, D (*a*). Now the vertical angles EFB, DFC are equal (*b*); the right angles BEF, CDF are equal (*c*), and the triangles EFB, DFC are equiangular (*d*); therefore FB : FE : : FC : FD, and alternately FB : FC : : FE : FD (*e*). Therefore also, the triangles EFD, BFC are equiangular: for their vertical angles are contained by proportional sides, as proved. Wherefore the angles FCB, EDF are equal: but EDF equals EAF as angles in the same segment (*f*); therefore EAF equals FCB: the vertical angles EFA, CFG are also equal (*b*); and the third angles AEF, CGF are equal (*g*). But AEF is a right angle;

Wherefore, CGF is a right angle; and AG, passing through the point F, is perpendicular to BC. Q. E. D

Recite (*a*) p. 31, 3; (*b*) p. 15, 1; (*c*) ax. 10, 1; (*d*) p. 4 and 6, 6; (*e*) p. 16, 5; (*f*) p. 21, 3; (*g*) p. 32, 1.

Cor. The triangles ADE, ABC are similar: for the triangles ABD, ACE, having right angles at D and E, and the angle A common, are equiangular, and have BA to AD as CA to AE; and alternately BA to CA as AD to AE. Therefore the two triangles BAC, DAE have the angle A common; and the sides about it proportionals; therefore they are equiangular (6, 6), and similar: and the rectangles BA×AF CA×AD are equal.

END OF THE SIX BOOKS.

# ELEMENTS

OF

# PLANE TRIGONOMETRY.

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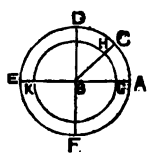
PLANE TRIGONOMETRY teaches the measurement of distance and elevation by means of angles. The principles of it are all contained in the preceding six Books.

Lemma 1. An angle at the centre of a circle is to four right angles, as the arc on which it stands is to the whole circumference.




Given ABC, a central angle, and AC the arc which measures it (*a*); the angle ABC is to four right angles, as the arc AC is to the whole circumference ACF.

Produce AB to meet the circle in E; and at right angles to AE draw DBF.

Then, because ABC, ABD are central angles; and AC, AD their arcs;—ABC : ABD : : AC : AD (*b*): moreover ABC : 4ABD : : AC : 4AD (*c*). But ABD is a right angle, and AD is the arc which measures it: therefore the angle ABC is to 4 right angles as the arc AC is to the whole circumference.

Recite (*a*) def. 1, trig.; (*b*) p. 33 of b. 6; (*c*) def. 3, and p. 4 of b. 5.

Cor. As the arcs AC and GH measure the same angle; AC is to the circumference ACE, as GH is to the circumference GHK: therefore equal angles at the centres of different circles stand on arcs which have the same ratio to their circumferences.

## *Definitions.*

1. The *measure* of an angle is the arc of the circle on which it stands: thus, the arc AC is the measure of the angle ABC.

2. *Degrees, minutes, seconds,* &c., are the terms of circle measure. Example. Every circle great or small is divided into 360 degrees, each degree into 60 minutes, each minute into 60 seconds. Hence an angle and its arc contain the same number of degrees, &c.

3. The *complement* of an angle is its defect from a right angle.

4. The *supplement* of an angle is its defect from a semicircle.

5. The *sine* of an angle is a straight line drawn in the bosom

(*sinus*) of the arc, from one end of it, perpending upon the diameter which meets the other end,—as CD. The greatest sine is that of a quadrant or 90 degrees; which is equal to the radius.

6. The *tangent* is a straight line touching the arc, parallel to the sine, and intercepted by the two sides which contain the angle,—as AE. The tangent of 45 degrees is equal to the radius.

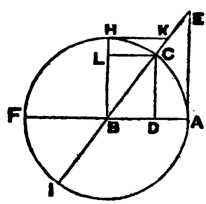
7. The *secant* is one of the sides which contain the angle, produced to meet the tangent, through the point in which the sine meets the arc,—as BE.

8. The *cosine*, *cotangent*, *and cosecant*, are the sine, tangent and secant, of the complement of the arc or angle.

9. The sine, tangent and secant of an arc, or angle, are also the sine, tangent and secant of its supplement.

10. The *subsine*, or versed sine, is a segment of the diameter intercepted between the sine and tangent,—as AD.

11. The subtense*, or hypothenuse, is the side opposite to a right angle.

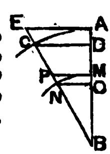



Note. It appears from the diagram, that any one of the lines, BA, BC, BH, &c., is a radius; that CD or BL is a sine; CL or BD a cosine; AE a tangent; HK a cotangent; BE a secant, and BK a cosecant. Moreover, if the arc AC, and its complements CH were equal, the tangent, cotangent and radius, would also be equal; as also the sine and cosine.

These lines form several equiangular triangles, whose sides are therefore proportionals, viz:

1. Cosine is to sine as radius is to tangent:—
   Ex. BD : DC : : BA : AE.
2. Tangent is to radius as radius is to cotangent:—
   Ex. AE : AB : : BH : HK.
3. Cosine is to radius as radius is to secant:—
   Ex. BD : BC : : BA : BE.
4. Sine is to radius as radius is to cosecant:—
   Ex. CD : CB : : BH : BK.
5. Sine is to cosine as radius is to cotangent:—
   Ex. BL : LC : : BH : HK.

12. The sine, subsine,* tangent and secant of an arc, taken as the measure of an angle, is to the sine, subsine, tangent and secant of any other arc measuring the same angle, as the radius of the first arc is to the radius of the other.



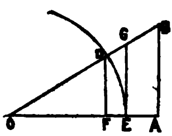

Given the arcs AC and MN, as measures of the angle ABC; and let CD be the sine, DA the subsine, AE the tangent, and BE the secant, of the arc AC. Also, let NO be the sine, OM the subsine, MP the tangent, and BP the secant, of the arc MN. The first lines are to the second, each to each, as the radius BC is to the radius BN.

* The terms *subsine* and *subtense* are more convenient than *versed sine* and *hypothenuse*.

Cor. If numerical tables of sines, subsines, tangents and secants, of certain angles to a given radius, be constructed, they will show the ratios of the sines, &c., of the same angles to any radius whatever. Such are called Trigonometrical Tables; in which the radius is represented as some term of the decimal series, 1, 10, 100, 1000, &c.

## *Propositions.*

1. Th. In a right-angled triangle, the subtense of the right angle is to either side, as radius is to the sine of the angle opposite to that side; and either of the sides is to the other, as radius is to the tangent of the angle opposite that other.




Given the triangle ABC, right angled at A; the subtense BC, and the sides CA, AB.

From the centre C, with any radius CD, describe the arc DE; draw DF at right angles to CE, and EG parallel to it, touching the arc in E, and meeting CB in G.

Then DF is the sine, EG the tangent, and CG the secant of the angle C, or of the arc DE.

The three triangles DFC, GEC, BAC, are equiangular: because the angles at F, E, A are right angles; and the angle C is common to all. Therefore,

1. CB : BA : : CD : DF;—but CD is the radius, and DF is the sine of the angle C (def. 5, trig.);

Therefore CB : BA : : R : sine of the angle C.

2. CA : AB : : CE : EG;—but CE is the radius, and EG is the tangent of the angle C (def. 6, trig.);

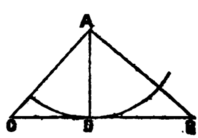
Therefore CA : AB : : R : tangent of the angle C.

Wherefore in a right angled triangle, &c. Q. E. D.

Cor. 1. Radius is to the secant as the side adjacent to the angle C is to the subtense. For CD, or CE : CG : : CA : CB.

Cor. 2. Let Radius $=1$; then the preceding analogies are as follows:

Sine of the angle C=BA divided by CB.
Tangent of the angle C=AB divided by AC.
Secant of the angle C=CB divided by AC.



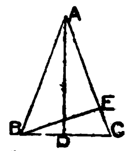

Cor. 3. If, in the triangle ABC, a perpendicular be drawn to the base BC; then (making AD radius),

AD : DC : : R : tan. CAD,
and AD : DB : : R : tan. BAD.

Cor. 4. Since, in a right angled triangle, the two acute angles are equal to the right angle, one of them is the complement of the other (def. 3 trig.), therefore any

position in a proportion, which the sine, tangent, or secant of one of them takes, may be given to the cosine, cotangent, or cosecant of the other.

## 2 Th. The sides of a plane triangle are to each other as the sines of the angles opposite to them.




In right angled triangles this proposition is obvious: for if the subtense be made radius, the sides are the sines of their opposite angles; and the radius is the sine of 90°, that is, of a right angle (*a*).

But in any oblique angled triangle, as ABC, any two sides, AC, BC, will have to each other the ratio of the sines of ABC, BAC, which are opposite to them.

From the angles A, B, draw AD, BE, perpendicular to BC, AC, produced if required: then, if AB be made radius, AD will become the sine of the angle B; and BE will become the sine of the angle A.

Since the triangles CDA, CEB, have right angles at D, E, and the angle C common, they are equiangular (*b*); therefore AC : AD : : BC : BE, and alternately AC : BC : : AD : BE (*c*): that is, the sides are as the sines of the angles opposite to them.

Wherefore, the sides of a plane triangle, &c. Q. E. D.

Recite (*a*) def. 5, P. T. (*b*) p. 32, 1, and p. 4, 6; (*c*) p. 16, 5.

Cor. Of two sides and two angles opposite to them, in a plane triangle, any three being given, the fourth is also given.

## Lemma 2. Of two unequal magnitudes, the greater equals half their sum more half their difference, the less equals half their sum less half their difference.

Given two magnitudes AB, BC, of which AB is the greater.


And since AC=AB+BC, the sum; make AE=BC: therefore AB—AE=EB, the difference of the two magnitudes. Bisect AC in D: then since AD=CD, and AE=CB; taking these equals from the former, the remainders ED, BD are equal; and EB, the difference is also bisected in D. It is therefore evident that AB=AD+DB, and BC=AD—DB; that is, the greater equals half the sum + half the difference, and the less equals half the sum — half the difference, as stated.

Otherwise: let s represent the sum, and d the difference.

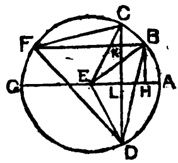
$AB+BC=s$, and $AB-BC=d$.

$2AB=s+d$, by addition; and $AB=\frac{1}{2}(s+d)$;

$2BC=s-d$, by subtraction; and $BC=\frac{1}{2}(s-d)$.

But $\frac{1}{2}(s+d)=\frac{1}{2}s+\frac{1}{2}d$; and $\frac{1}{2}(s-d)=\frac{1}{2}s-\frac{1}{2}d$.

3 Th. The sum of the sines of any two arcs of a circle, is to the difference of their sines, as the tangent of half the sum of the arcs is to the tangent of half their difference.




Let AB, AC be two arcs of a circle ABCD; E its centre; AEG the diameter which passes through A: it is to be proved that, the sin. AC+sin. AB : sin. AC—sin. AB : : tan. ½(AC+AB): tan. ½(AC—AB).

Draw BF parallel to AG, meeting the circle in F; draw the sines BH, CL perpendicular to AE; produce CL to D; join BE, CE, CF, DE, DF.

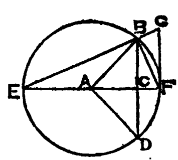
Since the perpendicular EL bisects CD and the arc CAD, DL equals CL, the sine of AC; the arcs AC, AD are also equal; and KL equals BH, the sine of AB. Therefore DK is the sum, and KC the difference of the sines; and DAB is the sum, and BC the difference of the arcs.

Now, in the triangle DFC, the perpendicular FK being made radius, DK : KC : : tan. DFK : tan. CFK.

But DK is the sum, and KC the difference of the sines: also DFK, CFK are halves of the central angles DEB, CEB, which are measured by the arcs DAB, BC, the sum and difference of the arcs AC, AB.

Therefore the sin. AC+sin. AB : sin. AC—sin. AB : : tan. ½(AC+AB) : tan. ½(AC—AB). Q. E. D.

4 Th. In any triangle, the sum of two sides is to their difference, as the tangent of half the sum of the angles at the base is to the tangent of half their difference.




Given the triangle ABC, the side AB being greater than AC; AB+AC : AB—AC : : tan. ½(ACB+ABC) : tan. ½(ACB—ABC).

With A as centre, and radius AB, describe the circle BEDF, meeting AC produced in E, F, and BC in D. Join EB, BF, AD; draw FG parallel to BC, meeting EB produced in G.

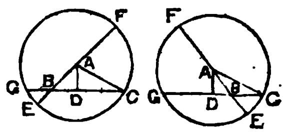
Then, because of the equal radii, EC=AB+AC, and CF=AB—AC: and since EBF is an angle in a semicircle, FB is perpendicular to EG; therefore EB, BG are the tangents of the angles BFE, BFG, or DBF, the alternate.

Moreover, the angles DBF, BFE are halves of the central angles DAF, EAB; also the exterior EAB=ACB+ABC, and the exterior ACB=CDA+CAD: but, because of the equal radii, ABD=ADB; therefore CAD=ACB—ABC. Therefore EB is the tangent of half ACB+ABC, and BG is the tangent of half ACB—ABC.

And because, in the triangle EFG, BC is drawn parallel to FG the base EC : CF : : EB : BG; that is, the sum of two sides is to their difference, as the tangent of half the sum of the angles at the base is to the tangent of half their difference.

Therefore, in any triangle, &c. Q. E. D.

5 Th. The base of a triangle is to the sum of both the sides, as the difference of the sides is to *the difference of the segments* of the base, made by a perpendicular falling upon it from the vertex, or *to the sum of the segments*, when the base must be produced to meet the perpendicular.




Given the triangle ABC, and AD, a perpendicular drawn from A to BC: and let AC be greater than AB.

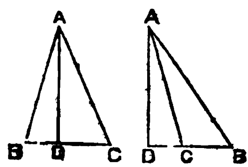
With A, as centre, and AC radius, describe a circle, meeting AB produced, in E, F, and CB in G.

Then, because the chords CG, EF, in a circle, cut one another, the rectangles EB×BF, GB×BC are equal (*a*); therefore BC : BF : : BE : BG. But BC is the base, BF=AC+AB, BE=AC—AB, and BG=CD—DB, or CD+DB, if the perpendicular fall on the base produced.

Wherefore, the base, &c. Q. E. D.

Recite (*a*) p. 35 of b. 3.

6 Th. In any triangle, the sum of two rectangles contained by the sides, is to the difference of the squares of those sides and the square of the base, as radius is to the cosine of the angle included by the two sides.




Given the triangle ABC; in which AC is the base, and AB, BC the sides: let AB be the radius;

then, $2AB\times BC : AB^2+BC^2-AC^2 : : R :$ cos. B.

Draw AD perpendicular to BC; then $2BC\times BD=AB^2+BC^2-AC^2$ (*a*).

But $BC\times BA : BC\times BD : : BA : BD$ (*b*) : : R : cos. B (*c*); therefore, also $2BC\times BA : 2BC\times BD : : R :$ cos. B (*b*).

Now $2BC\times BD$ is the difference between $AB^2+BC^2$ and $AC_2$.

Therefore $2AB\times BC : AB^2+BC^2-AC^2 : : R :$ cos. B.

Recite (*a*) p. 12, 13 of b. 2; (*b*) p. 15, 5; (*c*) cor. 4, p. 1, P. T.

Cor. Let radius = 1. Then, since

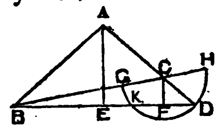
BA : BD : : R : cos. B, BD=BA×cos. B (*a*); and 2BC×BA× cos. B=2BC×BD (*b*); therefore when B is acute, 2BC×BA×cos. B=$BC^2+BA^2-AC^2$ (*c*). Add to both $AC^2$, then $AC^2+2BC\times BA\times$ cos. B=$BC^2+BA^2$. Take from both 2BC×BA×cos. B; then

$AC^2=BC^2+BA^2-2BC\times BA\times$ cos. B.

Wherefore, $AC=\sqrt{(BC^2+BA^2-2BC\times BA\times \text{cos. } B)}$; and the same result will appear by a similar process when the angle B is obtuse (*c*).

Recite (*a*) p. 16, 6; (*b*) p. 16, 6, and 15, 5; (*c*) p. 13, 12, of b. 2.

7 Th. If the base of a triangle be both increased and diminished by the difference of the two sides; four rectangles of the sides is to the rectangle of this sum and difference, as the square of the radius is to the square of the sine of half the angle contained by the sides.




In the triangle ABC, let AB be greater than AC, and make AD equal to AB; join BD, and draw AE, CF perpendicular to it; with C, as centre, and CD radius, describe the semicircle GDH, cutting BD in K, BC in G, and meeting BC produced in H.

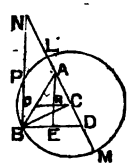
Because of the equal radii, BH=BC increased, and BG=BC diminished, by the difference of AB, AC. And since ABD is isosceles, and CF drawn from the centre, BD and KD are bisected in E, F. Wherefore DE—DF=EF; $\frac{1}{2}$(BD—DK)=$\frac{1}{2}$BK. Wherefore DE—DF=$\frac{1}{2}$(BD—DK). And because in the triangle DAE, CF is drawn parallel to the base, AC : AD : : EF : FD (2, 6); and rectangles of the same altitude being as their bases, $AC\times AD : AD^2 : : EF\times ED : ED^2$(1, 6); therefore $4AC\times AD : AD^2 : : 4EF\times ED : ED^2$, or alternately, $4AC\times AD : 4EF\times ED : : AD^2 : ED^2$.

But since 4EF=2BK, $4EF\times ED=2BK\times ED$, or $2ED\times BK=BD\times BK$, or $BH\times BG$ (cor. 36, 3); therefore $4AC\times AD : BH\times BG : : AD^2 : ED^2$. Now AD : ED : : R : sin. EAC=sin. $\frac{1}{2}$BAC (p. 1); therefore $AD^2 : ED^2 : : R^2 : (\text{sin. } \tfrac{1}{2}BAC)^2$. Therefore (11, 5), $4AC\times AD : BH\times BG : : R^2 : (\text{sin. } \tfrac{1}{2}BAC)^2$. But $4AC\times AD$ are 4 rectangles of the sides, and $BH\times BG$ is the rectangle of the base increased and diminished by the difference of the sides.

Wherefore, if the base, &c. Q. E. D.

Cor. $2\sqrt{AC\times AD} : \sqrt{BG\times BH} : : R : \text{sin. } \tfrac{1}{2}BAC$.

8 Th. Four rectangles of two sides of a triangle is to the rectangle of the whole perimeter and the excess of the two sides above the base, as the square of radius is to the square of the cosine of half the angle contained by those sides.




Given the triangle ABC, of which BC is the base, and AB greater than AC.

With C, as centre, and radius CB, describe the circle BLM, meeting AC produced, in L, M; produce AL, so that AN=AB; let AD=AB; join BD, BN; draw CO perpendicular to BN, and AE to BD.

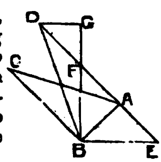
Now MN=AB+AC+CB, the perimeter; and LN=AB+AC—BC, the excess of the two sides above the base: and because BD is bisected in E, and DN in A, AE is parallel to BN, and perpendicular to BD; and the triangles DNB, DAE are equiangular. Wherefore, since DN=2AD, BN=2AE, and BP=2BO=2RE, also PN=2AB.

But since the triangles ABC, AED are equiangular, AC : AD : : AB : AE; and since rectangles of the same altitude are as their bases (1, 6) $AC\times AD : AD^2 :: AB\times AE : AE^2$; and alternately, $AC\times AD : AB\times AE :: AD^2 : AE^2$.

Moreover, $4AB\times AE = 2AB\times 2AE = NP\times NB = NM\times NL$. Therefore, $4AC\times AD : MN\times NL :: AD^2 : AE^2$; but AD : AE : : R : cos. DAE=cos. $\frac{1}{2}$BAC: therefore $4AC\times AD : MN\times NL :: R^2 : (\cos. \frac{1}{2}BAC)^2$; and AD=AB, and $MN\times NL$=the excess of the two sides above the base.

Wherefore, four rectangles, & Q. E. D.

9 Th. In any triangle, the less of two given sides is to the greater, as radius is to the tangent of an angle greater than half a right angle; also, radius is to the tangent of this excess, as the sum is to the difference of the sides; which sum and difference have the ratio of the tangent of half the sum to that of half the difference of the angles at the base.




1. In the triangle ABC, where AB is less than AC, AB : AC : : R : tangent of an angle exceeding half a right angle. Draw AD at right angles to BA, and make it equal to AC: join BD. If then BA be made radius, AD is the tangent of the angle ABD ($a$); which is the greater of the two acute angles, in the triangle ABD, and therefore exceeds half a right angle ($b$).

2. Radius is to the tangent of the excess just mentioned, as the sum is to the difference of the sides AB, AC. Produce DA to E; make AE, AF each equal to AB; join BE, and parallel to it draw DG; through the point F join BG.

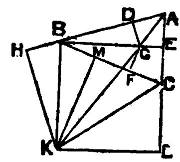
Then, because of the right angles at A, and that AF=AB, ABG is half a right angle; and DBG is the excess of ABD over that half. And because of the parallels DG, BE ($c$), and that EBG is a right angle, the alternate angle BGD is also a right angle. Then, if BG be made radius, DG is the tangent of the angle DBG, which is the excess, &c. Also, since G is a right angle, and the verticals at F are halves, the sides GF and GD are equal: moreover, ED is the sum and DF is the difference of AB, AC. Then, in the similar triangles FBE. FGD, FB : FE : : FG : FD, and alternately, FB : FG : : FE : FD; also, jointly BG : GF, or GD : : ED : DF; therefore, radius is to the tangent, &c., as stated.

3. Which sum and difference have the ratio of the tangent of half the sum to that of half the difference of the angles at the base. This is demonstrated in prop. 4.

Wherefore, in any triangle, &c. Q. E. D.

Recite ($a$) def. 6, P. T.; ($b$) P. 18, 1, and def. 3, P. T.; ($c$) p. 2, b. 6.

**10 Th.** If from half the perimeter of any triangle the base be taken, and also each of the sides; the rectangle of the half and its excess above the base, is to the rectangle of the other two excesses, as the square of the radiu is to the square of the tangent of half the angle contained by the sides.




Given the triangle ABC: bisect its angles at A and B, by the straight lines AG, BG; produce AG, AB, AC; bisect the exterior angles at B and C by the lines BK, CK; draw the perpendiculars GD, GE, GF, which are equal radii of the inscribed circle; draw also the perpendiculars KH, KM, KL; which are equal, because of the similar triangles.

Because the triangles AHK, ALK, are equiangular, and have AK common, AH is equal to AL; for the same reason BD is equal to BF, AD to AE, CE to CF, CM to CL, BH to BM.

And, because AD+BF+CE=AE+CF+BD=AE+CE+CM, taking away the parts shown to be equal, the remainders BF=BD=CM, ax. 3. Therefore AH+AL= perimeter, and AH=½ p. BC therefore equals HD, and AC equals BD; and the excess of ½ p. over BC=AD, over AB=HB, over AC=BD.

Hence the hypothesis is AH×AD : BH×BD : : R² : (T½BAC)²

Again, because the triangles BDG, KHB are similar, GD : BD : : BH : HK. Therefore GD×HK=BD×BH. But since AH : HK : : AD : DG. Therefore AH×AD : HK×DG : : AD×AD : DG×DG (p. 22 of b. 6); or AH×AD : BD×BH : : AD×AD : DG×DG. Now, if AD be made radius, DG will be the tangent of half the angle BAC.

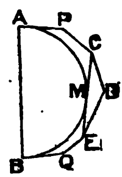
Wherefore, if from half, &c. Q. E. D.

# QUADRATURE OF THE CIRCLE.

*Definition.*

A convex line is any arc of a circle, or any polygonal line, which has no re-entrant angles, or inward inflections; and which a straight line cannot cut in more than two points.

**Lemma.** Any curve, or polygonal line, which passes round a convex line, from end to end, is longer than the convex line.




Let AMB be a convex line, enveloped from A to B, by the polygonal line APMQB: the distance from A or B to M, through P or Q, is longer than the part of the convex line from A or B to M; because every point of the convex line, from A to M, or from B to M, would fall within the triangle APM, or BQM, if the points M, P, A—M, Q, B, were joined by straight lines (*a*).

In like manner, it may be proved that the distance from P to Q, through CDE, is longer than the straight line PQ.

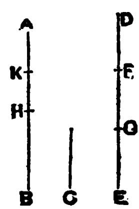
Therefore, any curve, or plolygonal line, which passes round, &c.

Recite (*a*) p. 21 of b. 1. Q. E. D.

Cor. 1. Hence the perimeter of any polygon inscribed in a circle is less than the circumference of the circle.

Cor. 2. And the perimeter of any polygon described about a circle exceeds the circumference of the circle.

**1 Th.** If from the greater of two unequal magnitudes its half be taken; and from the remainder its half; and so on; there will remain at length, a magnitude less than the less of the two given magnitudes.



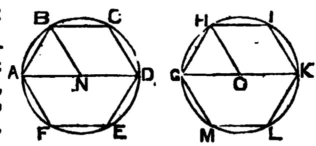

Let AB and C be two unequal magnitudes; of which AB is the greater: take DE a multiple of C, which is greater than AB; divide DE into parts DF, FG, GE, each equal to C. From AB take its half BH; and from the remainder take its half HK; until AB be divided into as many parts as DE.

Then since DE is greater than AB; and EG is not greater than the half of DE, but BH is the half of AB, the remainder DG is greater than the remainder AH. Again, since GF is not greater than the half of DG, but HK is the half of AH, the remainder DF is greater than the remainder AK.

Now DF is equal to C, the less of the two given magnitudes. Therefore, AK, the part left of the greater, is less than C. Q. E. D.

2 Th. Equilateral polygons of the same number of sides, inscribed in circles, are similar, and are to one another as the squares of the diameters of the circles.

Let ABCDEF and GHIKLM, be two equilateral polygons of the same number of sides, inscribed in the circles ABD, GHK: these two polygons are similar, and are to one another as the squares of the diameters AD, GK.



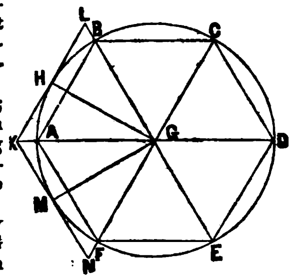

From the centres N, O, draw NB, OH. And because the sides of each polygon are all equal to one another, the arcs which they subtend in each polygon are also equal (p. 28 of b. 3); and the number of them in one circle is equal to the number of them in the other; therefore, whatever part any arc, AB, is of the circumference ABD, the same part is the arc GH, of the circumference GHK

But the angle ANB is the same part of four right angles, that the arc AB is of the circumference ABD (p. 33 of b. 6); and the angle GOH is the same part of four right angles, that the arc GH is of the circumference GHK (p. 33 of b. 6): therefore, the angles ANB, GOH, are each of them the same part of four right angles, and are therefore equal to one another. The isosceles triangles, ANB, GOH, are therefore equiangular, and the angle ABN equals the angle GHO. In the same manner, by joining NC, OI, it may be proved, that the angles NBC, OHI, are equal to one another, and to the angle ABN. Therefore, the whole angle ABC is equal to the whole GHI; and the same may be proved of the angles BCD, HIK; and so of the rest.

The polygons, ABCDEF and GHIKLM, are therefore equiangular to one another; and since they are equilateral, the sides about the equal angles are proportionals: these two polygons are therefore similar to one another (def. 1 of b. 6.) And because similar polygons are as the squares of their homologous sides, the polygon ABCDEF is to the polygon GHIKLM, as the square of AB to the square of GH: but because the triangles ANB, GOH are equiangular, the squares of AB, GH, are as the squares of AN, GO (p. 4 of b. 6); or as four times the square of AN to four times the square of GO (p. 15 of b. 5); that is, as the square of AD to the square of GK (cor. 2, p. 8, 2). Therefore also, the polygon ABCDEF is to the polygon GHIKLM, as the square of AD to the square of GK: and they have been shown to be similar.

Cor. Every equilateral polygon inscribed in a circle is also equiangular: for the isosceles triangles which have their common vertex in the centre, are all equal and similar; therefore, the angles at their bases are all equal, and the angles of the polygons are therefore also equal.

**8 P. The side of any equilateral polygon inscribed in a circle being given, to find the side of a polygon of the same number of sides described about the circle.**




Let ABCDEF be an equilateral polygon inscribed in the circle ABD; it is required to find the side of an equilateral polygon of as many sides described about the circle.

Find G the centre of the circle; join GA, GB; bisect the arc AB in H; and through H draw LK touching the circle, and meeting GA, GB produced in K and L; KL is the side of the polygon required.

Produce GF to N, so that GN may be equal to GL; join KN, and from G draw GM at right angles to KN, join also HG.

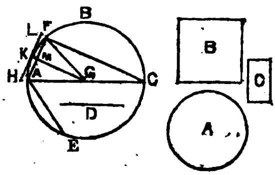
Because the arc AB is bisected in H, the angles AGH, BGH are equal (p. 27 of b. 3); and because KL touches the circle in H, the angles GHL, GHK are right angles (p. 18 of b. 3); therefore, the triangles HGK, HGL, have two angles in the one equal to two angles in the other, and the side GH is common to both; therefore they are equal (p. 26 of b. 1), and GL is equal to GK.

Again, in the triangles KGL, KGN, because GN is equal to GL, and GK common, and also the angles KGL, KGN equal; therefore the bases KL and KN are equal (p. 4 of b. 1). But because the triangle KGN is isosceles, the angles GKN and GNK are equal; and the angles GMK, GMN were made right angles; wherefore the triangles GMK, GMN have two angles in the one equal to two in the other, and the side GM is common to both; therefore they are equal (p. 26 of b. 1), and KN is bisected in M. But KN is equal to KL, therefore their halves KM, KH are also equal. Wherefore, in the triangles GKH, GKM, the two sides GK and KH are equal to the two GK and KM, each to each; the angles GKH, GKM are also equal; therefore GM is equal to GH (p. 4 of b. 1). Wherefore the point M is in the circumference of the circle; and because KMG is a right angle, KM touches the circle. Therefore KL is the side of an equilateral polygon described about the circle, of as many sides as the inscribed polygon ABCDEF.

Cor. 1. Because of the equal straight lines GL, GK, GN, if a circle be described from the centre G, through the points L, K, N, the polygon will be inscribed in that circle; and be similar to the polygon ABCDEF.

Cor. 2. The sides of the inscribed and described polygons, have to each other the ratio of the perpendiculars let fall from G upon AB and LK. Therefore because magnitudes have the ratio of their equimultiples (p. 15 of b. 5), the perimeters which are equimultiples of the sides of the polygons, are to each other as the perpendicular from the centre upon a side of the inscribed polygon is to the radius of the circle.

**4 Th. A circle being given, two similar polygons may be found, the one inscribed in the circle, the other described about it, whose difference shall be less than any given space.**




Let ABC be the given circle, and the square D any given space; a polygon may be inscribed in the circle ABC, and a similar one described about it, whose difference shall be less than the square of D.

In the circle ABC, apply the straight line AE, equal to D, and let AB be a quadrant. From AB take its half, from the residue its half, and so on, until the arc AF, be less than the arc AE (p. 1 of this). Find the centre G, draw the diameter AC, as also the straight lines AF, FG: bisecting the arc AF in K, join KG; and draw HL touching the circle in K, and meeting GA, GF produced in H and L: join CF.

Because the isosceles triangles HGL, AGF have the common angle AGF, they are equiangular (p. 6 of b. 6); therefore the angles, GHK, GAF are equal; but the angles GKH, CFA are also equal as right angles; therefore the triangles HGK, ACF, are likewise equiangular (p. 32 of b. 1).

And because the arc AF was found by taking from the arc AB its half, and from the residue its half, and so on; AF will be found a certain number of times exactly in AB, and therefore also in the whole circumference ABC. AF is therefore the side of an equilateral polygon inscribed in the circle ABC. Wherefore also, HL is the side of an equilateral polygon of the same number of sides described about the circle ABC (p. 3 of this).

Let the polygon described be called M, and the polygon inscribed be called N; and because they are similar, they are as the squares of the homologous sides HL and AF, (p. 20 of b. 6), that is (because the triangles HLG, AFG, are similar), as the square of HG to the square of AG, or its equal GK. But the triangles HGK, ACF were shown to be similar, and therefore the square of AC is to the square of CF, as the polygon M to the polygon N; and, by conversion, the square of AC is to its excess above the square of CF, viz. the square of AF (p. 47 of b. 1), as the polygon M is to its excess above the polygon N. But the square of AC, which is about the circle ABC, is greater than the regular octagon which is about the same circle; because the square envelopes the octagon (Lemma); and for the same reason, the polygon of eight sides is greater than one of sixteen, and so on: therefore the square of AC is greater than any polygon described about the circle by the continual bisection of the arc ABC: it is therefore greater than the polygon M.

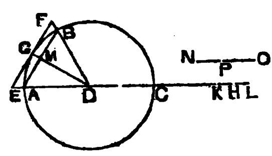
Now, it has been demonstrated, that the square of AC is to the square of AF, as the polygon M to the difference of the polygons; therefore, since the square of AC is greater than M, the square of AF is greater than the difference of the polygons, (p. 14 of b. 5). The difference of the polygons is therefore less than the square of AF; but

AF is less than D; therefore, the difference of the polygons is less than the square of D; that is, than the given space.

Cor. 1. Because the polygons M and N differ from each other more than either of them differs from the circle, the difference between each of them and the circle is less than the space given, the square of D. Therefore, however small a given space may be, a polygon may be inscribed in the circle, and another described about it, each of which shall differ from the circle by a space less than the given space.

Cor 2. The space B, which is greater than any polygon that can be inscribed in the circle A, and less than any polygon that can be described about it, is equal to the circle A. If not equal, let B exceed the circle A, by the space C. Then, because the polygons described about the circle A are all greater than D, by hypothesis; and because B is greater than A by the space C, therefore no polygon can be described about the circle A, that does not exceed it by a space greater than C; which is absurd. In the same manner, if B be less than A by the space C, it is shown that no polygon can be inscribed in the circle A, that is not less than A, by a space greater than C, which is also absurd. Therefore A and B are not unequal; that is, they are equal to one another.

5 Th. The area of any circle is equal to the rectangle contained by the radius and a straight line equal to half the circumference.




Let ABC be a circle, of which the centre is D, and the diameter AC; if in AC produced, there be taken AH equal to half the circumference, the area of the circle is equal to the rectangle contained by DA and AH.

Let AB be the side of any equilateral polygon inscribed in the circle ABC; bisect the arc AB in G, and through G draw EF touching the circle, and meeting DA produced in E, and DB produced in F, EF will be the side of an equilateral polygon described about the circle ABC (p. 3 of this). In AC produced take AK equal to half the perimeter of the polygon whose side is AB; and AL equal to half the perimeter of the polygon whose side is EF. Then AK will be less, and AL greater than the straight line AH. (Lemma of this). Now, because in the triangle EDF, DG is drawn perpendicular to the base, the triangle EDF is equal to the rectangle of DG and the half of EF, (p. 41 of b. 1), and as the same is true of all the triangles having their vertices in D, which make up the polygon described about the circle; therefore the whole polygon is equal to the rectangle contained by DG or DA and AL (p. 1 of b. 2). But AL is greater than AH; therefore the rectangle DA$\times$AL is greater than the rectangle DA$\times$AH; therefore the area of the circle ABC is less than that of any polygon described about it.

Again, the triangle ADB is equal to the rectangle contained by DM,

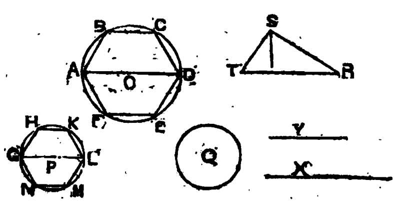
the perpendicular, and half the base AB; and it is therefore less than the rectangle contained by DG or DA and the half of AB. And as the same is true of all the other triangles having their vertices in D, which make up the inscribed polygon; therefore the whole of the inscribed polygon is less than the rectangle contained by DA and AK—half the perimeter of the polygon. Now the rectangle DA, AK is less than the rectangle DA, AH, and still less is the polygon whose side is AB. The rectangle DA, AH is therefore greater than any polygon inscribed in the circle ABC: and the same rectangle has been proved to be less than any polygon described about the circle; therefore the rectangle of DA, the radius, and AH, half the circumference, is equal to the area of the circle ABC. (Cor. 2 p. 4 of this.)

Cor. 1. Because DA : AH : : $DA^2$ : $DA \times AH$ (p. 1 of b. 6); and that $DA \times AH =$ area of the circle of which DA is the radius: therefore, as the radius of any circle is to the semi-circumference; or as the diameter to the circumference, so is the square of the radius to the area of the circle.

Cor. 2. Hence a polygon may be described about a circle, the perimeter of which shall exceed the circumference of the circle by a line that is less than any given line. Let NO be the given line. Take in NO the part NP less than its half, and less than AD; and let a polygon be described about the circle ABC, so that its excess above ABC may be less than the square of NP (cor. 1, p. 4 of this). Let the side of this polygon be EF. And since, as above, the circle is equal to the rectangle DA, AH, and the polygon to the rectangle DA, AL, the excess of the polygon above the circle is equal to the rectangle DA, HL; therefore the rectangle DA, HL is less than the square of NP; and therefore, since DA is greater than NP, HL is less than NP, and twice HL less than twice NP, wherefore twice HL is still less than NO. But HL is the difference between half the perimeter of the polygon whose side is EF, and the semi-circumference of the circle; therefore twice HL is the difference between the perimeter of the polygon and the circumference of the circle (p. 5 of b. 5); which is therefore less than NO.

Cor. 3. Hence also a polygon may be inscribed in a circle, such that the excess of the circumference above the perimeter of the polygon, may be less than any given line. This may be proved like the preceding.

## 6 Th. The areas of circles are in the duplicate ratio; or as the squares of their diameters.

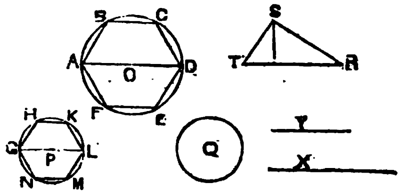



Let ABD and GHL be two circles, of which the diameters are AD and GL; the circle ABD is to the circle GHL as the square of AD to the square of GL.

Let ABCDEF and GHKLMN be two equilateral polygons

of the same number of sides inscribed in the circles ABD, GHL; and let Q be such a space, that the square of AD is to the square of GL as the circle ABD is to the space Q.

Because the polygons ABCDEF and GHKLMN are equilateral and of the same number of sides they are similar (p. 2 of this), and their areas are as the squares of the diameters of the circles in which they are inscribed. Therefore $AD^2 : GL^2 : :$ polygon ABCDEF : : polygon GHKLMN;—but $AD^2 : GL^2 : :$ circle ABD : Q. Now the circle ABD is greater than the polygon ABCDEF; therefore Q is greater than the polygon GHKLMN (p. 14 of b. 5); that is, Q is greater than any polygon inscribed in the circle GHL.




In the same manner it is demonstrated, that Q is less than any polygon described about the circle GHL; wherefore the space Q is equal to the circle GHL (cor. 2, p. 4 of this).

Now, by the hypothesis, the circle ABD is to the space Q as the square of AD to the square of GL; therefore the circle ABD is to the circle GHL, as the square of AD is to the square of GL.

Cor. 1. Hence the circumferences of circles are to each other as their diameters.

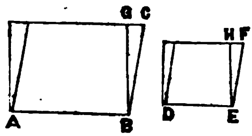
Let the straight line X be equal to half the circumference of the circle ABD, and the straight line Y be equal to half the circumference of the circle GHL: and because the rectangle $AO \times X$ and $GP \times Y$ are equal to the circles ABD, GHL (p. 5 of this); therefore $AO \times X : GP \times Y : : AD^2 : GL^2$, and as $AO^2 : GP^2$—and alternately, $AO \times X : AO^2 : : GP \times Y : GP^2$. Whence, because rectangles that have equal altitudes are as their bases (p. 1 of b. 6), X is to AO as Y is to GP, and alternately $X : Y : : AO : GP$. Wherefore, taking the doubles of each, the circumference ABD is to the circumference GHL, as the diameter AD to the diameter GL.

Cor. 2. The circle that is upon the side of a triangle opposite the right angle, is equal to the two circles described upon the sides containing the right angle.

For the circle described upon SR is to the circle described upon RT as the square of SR to the square of RT; and the circle described upon TS is to the circle described upon RT, as the square of ST to the square of RT.

Wherefore, the circles described upon SR and on ST are to the circle described on RT, as the squares of SR and of ST to the square of RT (p. 24 of b. 5). But the squares of RS and ST are equal to the square of RT (p. 47 of b. 1); therefore the circles described on RS and ST are equal to the circle described on RT.

7 Th. Equiangular parallelograms are to one another as the products of the numbers proportional to their sides.

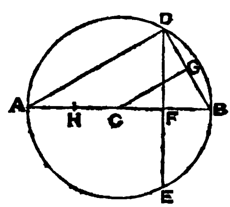



Dem. Because parallelograms upon the same base, or upon equal bases, and between the same parallels, are equal to each other ($a$); therefore the parallelograms AC and DF may be reduced to equivalent rectangles, as AG and DH. Now AG is the product of AB×BG ($b$), and DH is the product of DE×EH. Moreover BG is to BC as EH is to EF ($c$); and alternately, BG : EH : : BC : EF ($d$). Therefore AG : DH : : AB×BG : DE×EH; that is, AC : DF : : AB×BC : DE×EF.

Wherefore, equiangular parallelograms, &c. Q. E. D.

Recite ($a$) p. 35, of b. 1; ($b$) def. 3 of b. 2;
($c$) p. 4 of b. 6; ($d$) p. 16 of b. 5.

8 Th. A perpendicular drawn from the centre of a circle to the chord of any arc, is a mean proportional between half the radius and the straight line composed of the radius and its excess above the subsine of the arc: and the chord of the arc is a mean proportional between the diameter and the subsine.



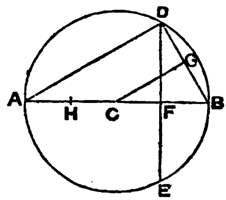

Let C be the centre of the circle ABD; let DB be any arc, and DBE twice the arc; draw the chords DB, DE; also CG and CF at right angles to DB, DE; produce CF to meet the circumference in B and A; bisect AC in H; join AD. BF is the subsine of the arc DB. (P. T. def. 10) Then AH : CG : : CG : AF; also, AB : BD : : BD : BF.

ADB is a right angle, being in a semicircle; and CGB is also a right angle: the triangles ADB, CGB, are therefore equiangular; and AB : AD : : CB : CG (p. 4 of b. 6); or, alternately, AB : CB : : AD : CG. And because AB is twice CB, AD is twice CG; and $AD^2=4CG^2$.

But because of the perpendicular DF, from the right angle, on AB; therefore AB : AD : : AD : AF (p. 8 of b. 6). Hence $AD^2=AB\times AF$ (p. 17 of b. 6); or, since AB=4AH, $AD^2=4AH\times AF=4CG^2$. Therefore $CG^2=AH\times AF$, and CG is therefore a mean proportional between AH and AF; that is, half the radius and the excess of the radius above the subsine BF.

Again, the chord BD is a mean proportional between the diameter and the subsine of the arc: for, (by p. 8 of b. 6,) AB : BD : : BD : BF.

**9 Th.** The circumference of a circle exceeds three times its diameter, by a line less than ten of the parts of which the diameter contains seventy; but greater than ten of the parts of which the diameter contains seventy-one.




Let C be the centre of the circle ABD, and AB its diameter; the circumference exceeds three times AB by a line less than 10-70ths, or 1-7th part of AB, but greater than 10-71st parts of AB.

Apply, in the circle, the chord BD, equal to the radius, or the side of an inscribed hexagon (p. 15 of b. 4); draw DF perpendicular on BC, and produce it to E; draw CG at right angles to DB; produce BC to A; bisect AC in H, and join AD.

The arcs BD, BE contain each 1-6th, and DBE 1-3d of the circumference; and (p. 8 of this) AH : CG : : CG : AF. Now, because the triangle BDC is equilateral, DF bisects BC in F. Therefore, if AC or BC$=1000$, AH$=500$, CF$=500$, and AF$=1500$; and as CG is a mean proportional between AH and AF, $CG^2=AH\times AF=500\times 1500=750000$. Wherefore, $CG=\sqrt{750000}=866.0254+$. Hence $AC+CG=1866.0254+$.

Now, as CG is the perpendicular from C on the chord of 1-6th of the circumference, if P be the perpendicular from C on the chord of 1-12th of the circumference, P will be a mean proportional between AH and AC+CG; and $P^2=AH\times(AC+CG)=500\times 1866.0254+=933012.7+$. Wherefore $P=\sqrt{933012.7+}=965.9258+$. Hence $AC+P=1965.9258+$.

Again, if Q be the perpendicular from C on the chord of 1-24th of the circumference, Q will be a mean proportional between AH and AC + P; and $Q^2=AH\times(AC+P)=500\times 1965.9258+=982962.9$. Wherefore $Q=\sqrt{982962.9}=991.4449+$. Hence $AC+Q=1991.4449+$.

Likewise, if S be the perpendicular from C on the chord of 1-48th of the circumference, S will be a mean proportional between AH and AC+Q; and $S^2=AH\times(AC+Q)=500\times 1991.4449+=995722.45+$. Wherefore $S=\sqrt{995722.45+}=997.8589+$. Hence $AC+S=1997.8589+$.

Lastly, if T be the perpendicular from C on the chord of 1-96th of the circumference, T will be a mean proportional between AH and AC+S; and $T^2=AH\times(AC+S)=500\times 1997.8589+=998929.45+$. Wherefore $T=\sqrt{998929.45+}=999.46458+$.

Thus, the perpendicular from the centre on the chord of 1-96th part of the circumference, exceeds 999.46458 of those parts of which the radius contains 1000.

But (p. 8 of this), the chord of 1-96th part of the circumference is a mean proportional between the diameter and the excess of the radius above the perpendicular from C on the chord of 1-48th part of the circumference.

Therefore the square of the chord of 1-96th part$=AB\times(AC-S)=$

$2000\times(1000-997.8589+)=2000\times2.1411-=4282.2-$ and the chord itself $=\sqrt{4282.2-}=65.4386-$; which is the side of a regular polygon of 96 sides, inscribed in the circle; and the perimeter of the polygon is equal to $96\times65.4386-=6282.1056-$.

Let the perimeter of the circumscribed polygon of 96 sides be M, then T : AC : : 6282.1056— : M; that is (since T=999.46458+, as already shown), 999.46458+ : 1000 : : 6282.1056— : M; if then N be such that 999.46458 : 1000 : : 6282.1056— : N; *ex æquo perturbato* 999.46458+ : 999.46458 : : N : M; and since the first is greater than the second, the third is greater than the fourth, or N is greater than M.

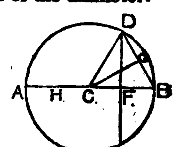
Now, if a fourth proportional be found to 999.46458, 1000 and 6282.1056, viz. 6285.461—, then because

999.46458 : 1000 : : 6282.1056 : 6285.461—, and as before,
999.46458 : 1000 : : 6282.1056— : N; therefore
6282.1056 : 6282.1056— : : 6285.461— : N;

and as the first of these proportionals is greater than the second, the third is greater than the fourth, viz. 6285.461—is greater than N. But N was shown to be greater than M; therefore 6285.461 is still greater than M, the perimeter of a polygon of 96 sides described about the circle; that is, the perimeter of that polygon is less than 6285.461; but the circumference of the circle is less than the perimeter of the polygon, and still less than 6285.461. Wherefore the circumference of a circle is less than 6285.461 of those parts of which the radius contains 1000.

The diameter, therefore, has to the circumference a less ratio (p. 8 of b. 5), than 2000 has to 6285.461, or than 1000 has to 3142.7305: but the ratio of 7 to 22 is greater than the ratio of 1000 to 3142.7305: therefore the diameter has a less ratio to the circumference than 7 has to 22; or the circumference is less than 22 of the parts of which the diameter contains 7.

The circumference exceeds 3 and 10-71st parts of the diameter.




It has been proved (p. 8 of this), that the perpendicular from the centre on the chord of an arc, is a mean proportional between half the radius and the sum of the radius and a perpendicular from the centre on the chord of twice the arc.

CG, P, Q, S and T were used, in this order, as perpendiculars on the chords of 1-6th, 1-12th, 1-24th, 1-48th and 1-96th part of the circumference.

$CG^2$, was made equal to $AH\times AF=500\times1500=750000$, and $CG=866.02545-$: therefore $AC+CG=1866.02545-$.

$P^2$. was made equal to $AH\times(AC+CG)=500\times1866.02545-$, and $P=965.92585-$: therefore $AC+P=1965.92585-$.

$Q^2$, was made equal to $AH\times(AC+P)=500\times1965.92585-$, and $Q=991.44495-$: therefore $AC+Q=1991.44495-$.

$S.^2$ was made equal to $AH\times(AC+Q)=500\times1991.44495-$, and $S=997.85895-$: therefore $AC+S=1997.85895$.

It is also proved (p. 8 of this), that the chord of the arc is a mean proportional between the diameter and the excess of the radius above the perpendicular: therefore (the chord of $1\text{-}96^2=AB\times(AC-S)=2000\times2.14105+=4282.1+$, and the chord $=65.4377+$. Now this

chord is the side of an inscribed polygon of 96 sides, the perimeter of which is 96×65.4377+=6282.019+. But the circumferenec of the circle exceeds the perimeter of the inscribed polygon; therefore the circumference exceeds 6282.019+ of those parts of which the radius contains 1000; or 3141.0095 of those parts of which the radius contains 500, or the diameter contains 1000. Now 1000 has a greater ratio to 3141.0095 than 1 has to 3 10-71st parts; that is, the excess of the circumference above three times the diameter is greater than 10 of those parts of which the diameter contains 71.

Cor. 1. Hence the diameter of a circle being given, the circum ference may be found nearly, by making as 7 to 22, so the given diameter to a fourth proportional, which shall be greater than the circumference. And, if as 1 to 3 10-71, or 71 to 223, so the given diameter to a fourth proportional, this will be nearly equal to the circumference, but will be less than it.

Cor. 2. Because the difference between 1-7th and 10-71st is 1-497th, therefore the lines found by these proportionals differ by 1-497th of the diameter; therefore the difference of either of them from the circumference must be less than the 497th part of the diameter.

Cor. 3. As 7 to 22 so the square of the radius to the area of the circle nearly.

For it has been shown (cor. 1 p. 5 of this), that the diameter of a circle is to its circumference as the square of the radius to the area of the circle; but the diameter is to the circumference nearly as 7 to 22, therefore the square of the radius is to the area of the circle nearly in that same ratio.

N. B. In the preceding calculations, for .461—read .47+.

# THE INTERSECTION OF PLANES.

## *Definitions.*

1. A straight line is perpendicular to a plane when it makes right angles with every straight line it meets in the plane.
2. A plane is perpendicular to a plane, when the straight lines drawn in one of the planes at right angles to their common section are perpendicular to the other plane.
3. The declination of a straight line from a plane is the acute angle of a right angled triangle opposite to the side perpendicular to the plane.
4. The declination of two intersecting planes is the angle made by two straight lines, one in each plane, which meet in the line of section, at right angles to that line.
5. Two planes have a declination equal to that of other two, when the angles of declination are equal.
6. A straight line is parallel to a plane, when it does not meet the plane, however produced.
7. Planes are parallel to each other, which do not meet, however far produced.
8 A solid angle is made by the meeting of more than two plane angles in the same point.

## *Propositions.*

1 Th. One part of a straight line cannot be in a plane and another part above it.

Recite Pos. 1; cor. def. 4, 1,

2 Th. Any three straight lines which meet each other not in the same point, are in one plane.

Recite def. 7, 1; p. 1 of this.

3 Th. If two planes cut one another, their common section is a straight line.

Recite def. 7, 1.

4 Th. If a straight line stand at right angles to each of two straight lines, at their point of intersection, it will also be at right angles to the plane in which those straight lines are.

Recite p 47 and 48 of b. 1; def. 1 of this.

5 Th. If three straight lines meet in a point, and a straight line stand at right angles to each of them in that point, these three straight lines shall be in the same plane.

Recite p. 3 and 4 of this.

6 Th. Two straight lines, which are at right angles to the same plane are parallel to each other.

Recite p. 47, 48 of b. 1; p. 2 and 5 of this; p. 28 of b. 1.

7 Th. If two straight lines be parallel, and one of them at right angles to a plane; the other is also at right angles to the same plane.

Recite p. 6 of this; ax. 11 of b. 1.

8 Th. Two straight lines, which are, each of them, parallel to the same straight line, though not both in the same plane with it, are parallel to each other.

Recite p. 4, 7, 6 of this.

9 Th. If two straight lines, which meet each other, be parallel to other two, though not in the same plane with the first two, the first two and the other two shall contain equal angles.

Recite p. 33 and 8 of b. 1; p. 4, 7, and def. 1 of this.

10 P. It is required to draw a straight line parallel to a plane.

Recite p. 12, 11, 31 of b. 1; p. 8 of this.

11 Th. From the same point in a plane there cannot be two straight lines at right angles to the plane, upon the same side of it. And there can be but one perpendicular to a plane from a point above it.

Recite p. 3, 6 of this.

12 Th. Planes to which the same straight line is perpendicular, are parallel to each other.

Recite def. 1 and 7 of this; p. 17 of b. 1.

13 Th. If two straight lines meeting one another, be parallel to two straight lines which also meet one another, but are not in the same plane with the first two, the plane which passes through the first two is parallel to the plane which passes through the others.

Recite p. 10, 8, 4, 12, and def. 1 of this; p. 31, 29 of b. 1.

14 Th. If two parallel planes be cut by another plane, their lines of section with it are parallels.

Recite Note def. 4 of b. 1.

15 Th. If two parallel planes be cut by a third plane, they have the same declination from that plane.

Recite p. 14, 4, 7, and def. 4 of this; p. 29 of b. 1.

16 Th. If two straight lines be cut by parallel planes, they must be cut in the same ratio.

**Recite p. 14 of this; p. 2 of b. 6; p. 11 of b. 5.**

17 Th. If a straight line be at right angles to a plane, every plane which passes through that line is at right angles to the first mentioned plane.

**Recite def. 1, 2, and p. 7 of this; p. 28 of b. 1.**

18 Th. If two planes cutting each other, be severally perpendicular to a third plane, their line of section is perpendicular to the same plane.

**Recite def. 2, 1, and p. 4 of this.**

19 P. Two straight lines, not in the same plane, being given in position, to draw a straight line perpendicular to them both.

**Recite p. 10, 13 and its cor., and def. 1 of this.**

20 Th. If a solid angle be contained by three plane angles, any two of those angles are greater than the third.

**Recite p. 23, 4, 20, 25 of b. 1.**

21 Th. The plane angles which contain any solid angle are together less than four right angles.

**Recite p. 20 of this p. 32 of b. 1.**

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## THE COMPARISON OF SOLIDS.

### *Definitions.*

1. A solid is that which has length, breadth and thickness.

2. Similar solid figures are such as are contained by the same number of similar planes, similarly situated, and declining equally from each other.

3. A pyramid is a solid figure contained by planes that are constituted between a plane and a point above it, in which they meet.

4. A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar and parallel to each other; and the others are parallelograms.

5. A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

6. A cube is a solid figure contained by six equal squares.

7. A sphere is a solid figure described by the revolution of a semicircle about a stationary diameter.

8. The axis of a sphere is the stationary straight line about which the semicircle revolves.

9. The centre of a sphere is the middle point of the axis.

10. The diameter of a sphere is any straight line passing through the centre to the surface on either side.

11. A cone is a solid figure described by the revolution of a right angled triangle about one of the sides containing the right angle, which remains fixed.

12. The axis of a cone is the stationary straight line about which the triangle revolves.

13. The base of a cone is the circle described by the revolving side containing the right angle.

14. A cylinder is a solid figure described by the revolution of a rectangle about one of its sides, which is fixed.

15. The axis of a cylinder is the stationary side of the rectangle which revolves.

16. The bases of a cylinder are the two opposite circles described by the ends of the rectangle.

17. Similar cones and cylinders are those which have their axes, and the diameter of their bases proportionals.

## *Propositions.*

1 Th. If two solids be contained by the same number of equal and similar planes, similarly situated; and if the declination of any two contiguous planes in the one solid be the same with the declination of the two equal and similarly situated planes in the other, the two solids are equal and similar.

Recite ax. 8 of b. 1.

2 Th. If a solid be contained by six planes, two and two of which are parallel, the opposite planes are equal and similar parallelograms.

Recite p. 14, 9, last article, and p. 4 of b. 1.

3 Th. If a solid parallelopiped be cut by a plane parallel to two of its opposite planes, it will be divided into two solids which will be to one another as the bases.

Recite p. 36, 1, and def. 1, 6; p. 2, 1 of this; p. 15 of last art., def. 5 of b. 5.

4 Th. If a solid parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes, it will be cut into two equal prisms.

Recite p. 8, 14, 15 last art., p. 34 of b. 1; p. 2, 1 of this.

5 Th. Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which

are terminated in the same straight lines in the plane opposite to the base, are equal to one another.

**Recite p. 34, 38, 36 of b. 1; p. 15 last art., p. 2, 1 of this.**

6 Th. Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines in the plane opposite to the base, are equal to one another.

**Recite def. 5 of this; p. 5 of the last article.**

7 Th. Solid parallelopipeds, which are upon equal bases, and of the same altitude, are equal to one another.

**Recite p. 11 last art., p. 14, 35 of b. 1; p. 7, 9 of b. 5; p. 3, 5, 6 of this.**

8 Th. Solid parallelopipeds which have the same altitude, are to each other as their bases.

**Recite p. 45 of b. 1; p. 7, 3, 4 of this; p. 4 of b. 5.**

9 Th. Solid parallelopipeds are to each other in the ratio composed of the ratios of the areas of their bases and of their altitudes.

**Recite def. 5, and p. 8, 7 of this; def. 10, and p. F, E of b. 5; p. 44 of b. 1; p. 12, 1 of b. 6, and cor. 2, p. 8 of this.**

10 Th. Solid parallelopipeds which have their bases and altitudes reciprocally proportional, are equal; and parallelopipeds which are equal, have their bases and altitudes reciprocally proportional.

**Recite p. 11, 9, and def. 10 of b. 5; p. 9 of this.**

11 Th. Similar solid parallelopipeds are to each other in the triplicate ratio of their homologous sides.

**Recite def. 2 of this; def. 1, and p. 23 of b. 6; def. 9 of b. 5.**

12 Th. If two triangular pyramids, which have equal bases and altitudes, be cut by planes that are parallel to the bases, and at equal distances from them, the sections are equal to each other.

**Recite p. 14, 9, 16 of the last art., p. 18, 1 of b. 5; p. 22 of b. 6.**

13 Th. A series of prisms of the same altitude may be described about any pyramid, such that the sum of the prisms shall exceed the pyramid by a solid less than any given solid.

**Recite cor. 1, p. 12, def. 4, and cor. 1, p. 8 of this.**

14 Th. Pyramids that have equal bases and altitudes are equal to one another.

**Recite p. 13, 12, and cor. 1, p. 8 of this.**

15 Th. Every prism having a triangular base may be divided into three pyramids that have triangular bases, and that are equal to one another.

**Recite p. 34 of b. 1; p. 14, and cor. 1, p. 8 of this.**

16 Th. If from any point in the circumference of the base of a cylinder, a straight line be drawn perpendicular to the plane of the base; that line will be wholly in the cylindric superficies.

**Recite def. 14 of this, p. 6, 14 last article.**

17. Th. A cylinder and a parallelopiped having equal bases and altitudes, are equal to each other.

**Recite cor. 1, p. 4 Qu. Cir., p. 16, and cor. 2, p. 8 of this.**

18 Th. If a cone and cylinder have the same base and the same altitude, the cone is the third part of the cylinder.

**Recite p. 4 Qu. Cir., p. 15 of this.**

19 Th. If a hemisphere and a cone have equal bases and altitudes, a series of cylinders may be inscribed in the hemisphere, and another series may be described about the cone, having all the same altitudes with one another, and such that their sum shall differ from the sum of the hemisphere and the cone, by a solid less than any given solid.

**Recite def. 7, 11, 14, and p. 13 of this.**

20 Th. The same things being supposed as in the last proposition, the sum of all the cylinders inscribed in the hemisphere, and described about the cone, is equal to a cylinder, having the same base and altitude with the hemisphere.

**Recite cor. 2, p. 6 Qu. Cir.**

21 Th. Every sphere is two-thirds of the circumscribing cylinder.

**Recite p. 18, 19, 20 of this.**

THE END.

*To the Officers and Teachers of the Public Schools, and Principals of Academies.*

GENTLEMEN,—The subscriber has the honor of submitting to your consideration, a cheap, simple and practicable plan of teaching the entire elements of geometry to very young learners, without changing the approved order of Euclid.

These elements belong to the education of the operative classes, whose business is to contend with the resistance of matter, not less than to the liberally educated: for, in the one case, they describe the co-operative tendencies of natural action; and in the other, they furnish the best model of reasoning on every subject of human knowledge.

Why, it may be asked, is geometry so slightly passed over, as though it were subordinate among scholastic studies? The reason is briefly this: the elements and the arguments are locked in the same form; and the alternative presented to the schools is—*all*, or *none*. Hence it is that geometry is so little known—so little appreciated as a branch of popular instruction; that while other branches are cultivated to the summit of perfection, and often well over it, to continue the simile, in adapting them to all ages and tastes, this subject has been neglected. The book-makers have not attempted to reduce it to a condition which might secure the general diffusion of these elements in the schools.

The plan now respectfully submitted, proposes to teach geometry as grammar or geography is taught; that is, by questions: to which may be added the peculiar advantage of drawing the diagrams on slates. This, it is believed, will remove all the difficulties connected with this essentially important study; and the introduction of these elements will improve the character of the schools.

The subscriber has pursued this subject for many years, conscientiously, and to his own disadvantage. Should it, therefore, seem proper to your Boards to adopt his plan, he will feel gratified in the assurance that he has not labored in vain to extend the basis of useful knowledge.

The publishers are prepared to supply the schools on moderate terms.

With very high respect,
Your obedient servant,
D. M'CURDY.

## RECOMMENDATIONS OF THE CHART AND FIRST LESSONS.

*Philadelphia, January* 31, 1845.

Mr. D. M'Curdy,—Dear Sir:—I have examined the Chart of Geometrical Diagrams, and Book of First Lessons, which you were so kind as to submit to my inspection. As the main design appears to be to familiarize pupils with the several propositions of Euclid's elements—to fix their demonstrations on the mind, by making the reference to the preceding propositions easy and expeditious;—and save the time usually spent in drawing figures *repeatedly*, in the manner usually prevalent; I conceive that it must commend itself strongly to the attention of teachers having large numbers of students in this branch, whose time they would economize, and whose acquisitions they would render permanent.

Respectfully yours,
WALTER R. JOHNSON,
Prof. Math. Franklin Institute, Phila.

The following gentlemen have expressed their concurrence in the recommendation of Professor Johnson, namely:

SETH SMITH, Teacher of Friends' Public School, Green St. Phila.
JAMES RHOADS, Principal of N. W. Grammar School.
A. T. W. WRIGHT, Principal of the Model School, Philada.
NICH. H. MAGUIRE, Principal of Coates St. Grammar School.

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*Philadelphia, Feb.* 19, 1845.

I conceive that the Chart of the diagrams of Euclid, with the accompanying book of first Lessons, containing the propositions, &c. might be used to some extent in the Grammar Schools of Philadelphia, without detriment to the other branches of education taught therein; and that its introduction would conduce to the diffusion of the elements of Geometry among a class of the community whom the present provisions never reach.

JOHN M. COLEMAN,
Principal of the New Market Street Grammar School, Phila.

*Concurrent:*
TIMOTHY CLOWES, L.L. D., Principal of Jefferson Grammar School.
P. A. CREGAR, Principal of the S. E. Grammar School, Phila.
WM. ROBERTS, Principal of the Moyamensing Grammar School.
W. J. KURTZ, Principal of Falls Public School.
ANDREW CROZIER, Principal of the Reed St. Gram. School.
SAML. F. WATSON, Principal of Catherine St. Gram. School.
CHARLES HOAG, Principal of the Lower Dublin Gram. School.

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I cordially concur in the opinion of Mr. Coleman, respecting the advantage which would result from the use of Mr. M'Curdy's plan of

teaching Geometry, in the Public Schools, of the City and County of Philadelphia; and indeed in all other schools of like grade.

JAMES M'CLUNE,
Principal of the Master St. Grammar School.

*Concurrent:*

JOSHUA RHOADS, M.D., Principal of Palmer St. Gram. School.
A. H. BROWN, Principal of Zane St. Grammar School.
W. W. WOOD, Principal of S. W. Grammar School.
W. H. PILE, Principal of N. E. Grammar School.
B. E. CHAMBERLIN, Principal of Buttonwood St. Gram. School.
W. S. CLEAVENGER, Principal of Locust St. Gram. School.

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*Central High School, Philadelphia, February* 24, 1845.

Having examined a Chart of Geometry, prepared by Mr. D. M'Curdy, containing the diagrams and propositions* of Euclid, in the order of Simson's and Playfair's editions, we are of opinion that the use of it, as "First Lessons," in schools, would conduce to the more general diffusion of Geometrical knowledge, and be very helpful to those who may want the time and facilities for a more liberal Mathematical education. The Chart appears to be correct and neatly executed.

E. OTIS KENDALL,
Prof. of Theoretical Math. and Astronomy.
WILLIAM VOGDES,
Professor of Practical Mathematics.

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*Lancaster, Pa. March* 1, 1845.

I concur very cordially with Professor Johnson of Philadelphia, in the recommendation of Mr. M'Curdy's Geometrical Chart, No. 1.

DANIEL KIRKWOOD,
Principal of the Male High School of Lancaster.

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I have examined the Chart of Geometry, No. 1, and feel warranted in adding a cordial concurrence with the Professors and Principals who have subscribed their names in testimony of its great utility.

JAMES DAMANT,
Principal of the Ladies' Seminary of Lancaster, Pa.

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The Committee on books of the Public Schools for Lancaster County, consisting of the Rev. Clergy of the City of Lancaster, passed a resolution on the day above mentioned, to recommend the adoption of the Geometrical Charts to the Board of Directors of which they are a Committee.

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*Columbia, Lancaster County, Pa. March* 4, 1845.

I have examined the Chart of Geometry prepared by Mr. D. M'Curdy, exhibiting the diagrams and propositions in the order of Simson's

* The propositions were attached to the first plate of the Chart, which are now contained in the book, called "First Lessons."

and Playfair's Euclid, and pronounce it superior to any work of the kind, that has come to my knowledge, for facilitating the study of those elements.

THOMAS W. SUMMERS,
Principal of the Columbia Academy.

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I am decidedly of opinion, that the Chart of Geometry, in the hands of judicious teachers, would be very useful in our Primary Schools, in giving to the young some knowledge of this important branch of science.

STEPHEN BOYER,
Principal of York Co. Academy.

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*March* 5, 1845.

I cordially concur with the Rev. Mr. Boyer, in the opinion that Mr. M'Curdy's plan of teaching Geometry could be used to great advantage in our Primary Schools of the South Ward, also that it would facilitate the study more than the usual books, and with less expense.

WILLIAM R. STOUCH,
Teacher, York, Penn.

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*Washington, March* 18, 1845.

Dear Sir:—I have examined your Geometrical Charts, and am much pleased with the plan and arrangement; and believing it to be the very best system to instruct the youthful mind in the principles of this invaluable science, I will introduce a bill into the Councils next Monday, to make appropriation for the purchase of a set of those Charts, for each of our Public Schools. The simple fact of having those figures before the eyes of the children familiarizes their minds with them; and the questions they will naturally ask in relation to them, will so impress them on their minds, that the study itself in more advanced years will be comparatively easy. In fact I consider the pictorial plan of instruction as much of an improvement on the old system, as steam and magnetism are on the plans of locomotion.

Very truly, your friend,
JOHN WILSON,
Alderman of the Second Ward, Washington.

D. M'Curdy, Esq., Washington, D. C.

The Mayor of Washington certifies that on the 2d of June, the Board of Trustees ordered the purchase of a set of the Charts, &c., for each of the Public Schools.

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*Respectfully submitted to the Secretary of War.*

The course of practical Geometry, after the method of Mr. M'Curdy, may be introduced with advantage amongst the non-commissioned officers and soldiers of the army.

We have schools at many of our military posts, where the men receive gratuitous instruction from their officers, in the ordinary branches

of an English education; and I would regard the addition of Mr. M'Curdy's Geometrical Charts, as an important *first step*, towards bringing into our service, a system of instruction kindred to that adopted for the non-commissioned officers of the British army, and with so much success, at the Royal Military School of Woolwich.

WINFIELD SCOTT.
May 31, 1845.

A true copy. H. L. SCOTT, Aide-de-camp.

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The Chart and First Lessons have been introduced into the Union Public Schools, and several private Academies in Wilmington, Del. Also in Georgetown, D. C.; in Camden and Burlington, N. J.; in Frankford, Germantown and Bristol, Pennsylvania; and in several Academies in Philadelphia. And the Commissioners of Public Schools in the city of Baltimore, in August last, ordered the Charts and First Lessons to be used in the schools of that city.

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*New York, Oct. 18th*, 1845.

We have examined M'Curdy's Chart of the diagrams of Euclid, and the book of "First Lessons" connected therewith; and think his plan well adapted to the purpose of teaching large classes, and children of immature minds. Its use is a *desideratum* in the schools.

| | | |
|---|---|---|
| (Signed) | DAVID PATTERSON, M.D.<br>JOSEPH M'KEEN, A.M. | Teachers of the Male Normal* School. |
| | LEONARD HAZELTINE,<br>WILLIAM BELDEN, A.M. | Teach. of the Female Normal School. |

*Concurrent.* ISAAC F. BRAGG, Prin. Male High School.
JAMES N. M'ELLIGOTT, Prin. Mech. Soc. School.
MILTON C. TRACY, Prin. Mech. Inst. School.

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I have examined M'Curdy's "First Lessons and Chart;" and have no hesitation in stating my belief, that they will prove a valuable auxiliary to the acquisition of the important study of Geometry.

(Signed) JOHN W. KETCHAM, Prin. Pub. School No. 7.

*Concurrent.* JOHN PATTERSON, Prin. Public School No. 4.

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MR. FOULKE,—Dear Sir,—Mr. M'Curdy will show you a map to facilitate the study of Geometry, with a very concise little work on the subject, which *we*, the school officers, think well of. *If you approve of the same*, I authorize you to take one set of the maps and a dozen of the books, and send the bill to me. Yours, truly.

HENRY NICOLL.

Tuesday, Nov. 11, 1845.

Having hastily examined the work above referred to, I have no hesitation in saying, that I think the plan of the author well adapted to the

* In these Institutions teachers are prepared for the Public Schools.—ED.

purpose of teaching Geometry to large classes of pupils who are about entering upon the study of this important science; and of supplying a manifest defect now existing in our schools.

THOMAS FOULKE, Prin. W. School No. 14, N. Y.

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We have examined the Chart of Geometry and "Book of First Lessons," prepared by D. M'Curdy, and believe them to be well adapted to the purpose of teaching this science in public schools. In our opinion, a pupil, by using these charts, and aided by the teacher, will obtain a clearer idea of Geometry, besides saving time, than by any other method.

WM. KENNEDY, Prin. W. School No. 2, N. Y.
JOHN J. DOANE, Asst. W. School No. 2.

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The undersigned, members of the Book Committee of the Ward School Teachers' Association, having examined a chart of the diagrams of Euclid, connected with a book called "First Lessons," and prepared for public schools by Mr. D. M'Curdy, respectfully

REPORT:

That the design of said book and chart appears to be to supply a defect in the elements of a sound practical education, the existence of which your committee must readily admit. The cause of this defect, however, is not in the want of books on Geometry; but in the oppressive strictness in which the subject is usually presented to beginners. The plan now presented obviates this difficulty by omitting the demonstrations, and requiring learners to read the propositions, recite the proofs and draw the diagrams, as occasional exercises, on slates. Questions are also suggested to teachers in view of the book and chart, which, if followed out, must inevitably render the subject familiar to both teachers and pupils; and prepare them to demonstrate the propositions of Euclid with facility and success. Your committee therefore recommends this work to the patronage of the Association as worthy of its unanimous support, and offers the following resolution:—

*Resolved*, That this Association recommend M'Curdy's geometrical chart of the diagrams of Euclid in connection with the book of "First Lessons," with a view to their introduction into the Ward Schools of this city.

Signed, SENECA DURAND,
E. H. JENNEY,
JOHN WALSH,
EDWARD M'ILROY.

At a regular meeting of the Ward School Teachers' Association on the 17th Dec., 1845, the above report of the committee was approved and the resolution unanimously adopted.

JOSIAH RICH, President W. S. T. Asso.

## FURTHER DIRECTIONS FOR THE USE OF THE CHART OF GEOMETRY AND BOOK OF FIRST LESSONS.

*Respectfully submitted to the Officers and Teachers of the Public Schools.*

GENTLEMEN,—In reply to a question proposed, How are the Chart and First Lessons, to be used?—permit me to explain in a few words.

Each student requires a copy of the book; one chart will be sufficient for a large class. The students read the definitions, postulates, axioms and propositions, in order; and recite the proofs at each proposition, as directed. The teacher should not embarrass himself or his class, by exacting from them more than they can perform: he ought not to stop, in the first course, except to explain the meaning of terms. The object is to make the enunciations familiar to the students. The six books of Euclid and plane trigonometry occupy forty-three pages; these may be read and recited every six months, without any material abatement of other studies.

In the second course of reading, slates and pencils may be used:—in order to draw a straight line from one point to another—to produce the same in a straight line—and to describe a circle about a point with any radius. This is the permission sought in the postulates; and it embraces every thing that is to be done by hand.

Next, in connection with the definitions, the class draws straight lines, parallel, and at right and oblique angles to each other; also radii, diameters, and other chords in a circle;—triangles equilateral, isosceles and scalene—right angled and oblique;—squares and other rectangles, rhombuses, rhomboids, and a variety of trapezia: then they should make several angles at the same point, and read them distinctly by their proper letters. In short, the class should draw every line and figure spoken of in the definitions, and call it by its proper name: for then they will take an interest in it as their own production.

In view of the axioms, every relation should be expressed by appropriate lines, angles, or areas; illustrating each according to the tenor of it: this exercise will impress the use of these simple propositions upon the learner's memory, and be to him a pleasing introduction to the process of ratiocination: axioms are always invincible proofs.

The teacher will now direct the class to copy from the chart as many of the diagrams as he thinks proper for the day's lesson: he will examine whether they have done it well or ill; and ask, in relation to each diagram, what proposition it illustrates, and what proofs are adduced;—all of which the class will answer: and as they recite proof after proof, he will point to the proper diagram on the chart, and say—"This is the illustration—How does it apply to the case?" He cannot expect answers in all cases: but the question will elicit attention.

The definitions and axioms are properly quoted as proofs: for although they are not separately represented on the chart, they are found all over it. The postulates prove only this, that the geometer asks permission to draw such lines as he requires: the fact of his drawing them shows that he has permission.

Some of the elements are of more general use in demonstrations than others, and are more frequently quoted. Of such is the equality of radii;—of triangles identical (as in p. 4 of b. 1), and those upon the same or equal bases and between the same parallels; also of parallelograms in similar case;—of angles by position vertical and opposite, alternate, and in alternate segments of a circle made by a chord meeting a tangent, interior, exterior and adjacent;—of the opposite sides and angles of parallelograms;—of the square of the side subtending to the squares of the sides containing a right angle;—of the squares and rectangles upon lines divided equally and unequally, or bisected and produced;—of ratios; and of similar rectilineal figures. These equals are among the most useful of the elements, and will claim the learner's attention.

The object of the First Lessons is to furnish the *entire text, unencumbered with comment*, which has ever precluded the general diffusion of these useful elements. The chart furnishes the diagrams, as an auxiliary to the acquirement of the text, which is the great desideratum. They do not speak from experience who say that students cannot learn the enunciations without the demonstrations: their doubts, however, will be received as arguments only by the indolent, "who stand by the brook until it shall have discharged all its waters." On the other hand, no one will pretend to say, that the demonstrations can be effected without that familiarity with the propositions provided for in the Chart and First Lessons.

The time has arrived when the elements of geometry should obtain a place in the common education. Abstracts for the use of Surveyors, Navigators, Builders, &c., are insufficient: the qualification for an occupation, or office, should precede the exercise of it, and be the common portion of junior citizens. The old plan proves to be impracticable by the public destitution of this most useful knowledge: and it is by far too costly for large schools. The plan here proposed is accessible to every reader, and requires only the test of experiment. A book worth 25 cents, in the hands of willing teachers, will effect the general diffusion of these inestimable elements of natural science in the schools: not merely in view of the future occupations of the students; but as the basis of that general knowledge required by every citizen of the United States;—as the best remedy for the contagion of the turbid streams now inundating town and country under the name of literature; which robs youth of its time for improvement, and leads many into the imitation of vices, at least palliated if not approved.

In regard to the manner of using the Chart and First Lessons, it would seem unnecessary to say more. The preface to the book contains forms of questions applicable to the diagrams and propositions. We shall add the substance of the foregoing directions. If any difficulty remain, it must be that of invincible incredulity: for what can be more simple than to read the book and understand what you can of it;—to repeat the same and learn more and more;—to follow out the directions given and make your knowledge perfect?

The readings and recitations are all the text of Euclid, without comment;—the collective wisdom of the past; by which the remote parts of the earth have been discovered, rendered habitable, and adorned with cottages, cities and temples to the praise of Him who reigns.

## OLMSTED'S SCHOOL ASTRONOMY:

Containing the Elements of the Science, familiarly explained and illustrated. With the latest Discoveries. Adapted to the use of Schools and Academies, and of the general reader. By *Denison Olmsted, A.M.*, Professor of Natural Philosophy and Astronomy in Yale College. 12mo. Sheep. Price 75 cents.

This work having been used as a class book in many schools and academies, the publishers have been favoured with testimonials of approbation, among which are the following:—

*From Charles Henry Alden, Principal of the Philadelphia High School for Young Ladies.*

I have examined with great care "*Olmsted's Compendium of Astronomy,*" and have taken a highly intelligent class in my institution critically through it. We have long felt the want of a Text Book in this most interesting science, and the author of this merits the thanks of the profession for a Treatise so entirely methodical and lucid, and so admirably adapted to our more advanced classes. Judging of its merits by the interest evinced by my pupils, as well as from its intrinsic excellence, I cannot too strongly recommend its general adoption.

---

*From Samuel Jones, A. M., Principal of the Classical and Mathematical Institute, South Seventh Street, Phil.*

I am using "Olmsted's Compendium of Astronomy" in my school, and fully concur with the Rev. Mr. Alden in his opinion of the work.

---

*From Charles Dexter Cleveland, A. M., late Professor in Dickinson College, Carlisle, Pa.*

GENTLEMEN,

I have not vanity enough, I assure you, to suppose for a moment that any thing I can say will add to the fame of Professor Olmsted's Works on Natural Philosophy; but as you ask me my opinion of his "Compendium of Astronomy," I will say, that I intend to introduce it into my school, considering it the best work of the kind with which I am acquainted.

This work has also been very favourably noticed in various periodicals. The following discriminating remarks from the *New Haven Record*, are understood to be from the pen of an able and experienced teacher of Astronomy.

Olmsted's School Astronomy.—It is with peculiar pleasure we notice the appearance of this work, small in size, but containing more matter than many larger books. There is probably no instructor of much experience who has not felt serious inconvenience from the want of a proper text book in this department of science, as taught in our academies and higher schools. The treatise before us, however, is one which, after a careful perusal and the use of it as a text book, we can most cheerfully recommend as eminently adapted to supply the vacancy heretofore existing. Our author is particularly happy in the arrangement and division of the various subjects discussed; each occupying its appropriate place, involving no principle which has not been previously considered. He aims to fix in the mind the great principles of the science, first by stating them in the most concise and perspicuous terms, and then by lucid and familiar illustrations, without entering into an indiscriminate and detailed statement of a multiplicity of statistics, which only burden the memory and discourage the student.

The learner is early made acquainted with the use of the globes, which greatly assists him in understanding the causes of various phenomena which otherwise would be wholly unintelligible. The student in his progress finds himself in possession of the keys of knowledge, and is highly pleased and encouraged in being able to anticipate effects and assign their true causes, which before seemed a mystery. Indeed, we feel assured, that this compendium need only be known to competent and judicious instructors, in order to secure its cheerful introduction into our schools and academies as an important auxiliary to science. Those who have had the misfortune to toil for many a tedious hour in treasuring up insulated facts in this science, without a knowledge of the principles upon which they depend, and consequently without any knowledge of the subject, will be fully convinced, after a patient perusal of the work under consideration, that the study of Astronomy is not necessarily dry and uninteresting, but that the difficulty arises in a great measure from an erroneous method of studying the subject. A.

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1845 & 6.

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*Secretary's Office, Department of Common Schools.*

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